UAGS Lecture Notes 0

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These notes cover foundational material on topologies and rings that is assumed for the seminar.

1 Topologies

If I give you a set, then you can say which things are in it and which aren't, but not much more. In geometry, we want to consider shapes and spaces; these are objects which have additional structure. As a first step towards formalizing these additional properties of shapes, we might try to make rigorous the notion of *closeness*. This is what a topology does.

Definition 1.1. Let X be a set. A *topology* $\mathcal{T}$ on X is a collection of subsets of X, called the *open subsets*, which satisfies the following conditions:

1. $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$.
2. If $\{U_\alpha\}$ is an arbitrary collection with $U_\alpha \in \mathcal{T}$ for all α, then $\bigcup_\alpha U_\alpha \in \mathcal{T}$.
3. If U_1 and U_2 are in $\mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$.

Intuitively, given a point $p \in X$ and an open set U containing p, the fact that U is open means that it must contain all of the points "close" to p. By examining our space at different resolutions, we can understand closeness in different ways. At the coarsest resolution possible, we can just consider all of the points in our space to be close, so X should be in $\mathcal{T}$. Since there aren't any points in the empty set, it should be open trivially. If we have a collection of sets, all of which contain all of the points close to any of their members, then their union will also satisfy this property. Finally, if we're given two open sets U_1 and U_2, by increasing our resolution, we can see that for every member of $U_1 \cap U_2$, the set contains every point close to them.

Definition 1.2. A topological space is the data of a set X and a topology $\mathcal{T}$ on X.

Example 1.3. On any set X, we can just take $\mathcal{T} = \mathcal{P}(X)$ (so $\mathcal{T}$ contains all of the subsets of X). This is called the *discrete topology* on X. Consider the discrete topological space $(X, \mathcal{T})$. Since for any point $p \in X$, the set $\{p\}$ is in $\mathcal{T}$, we should think of all of the points of X as being very far separated from each other, since it is possible to examine X with a high enough resolution to pick out individual points.

Example 1.4. On the other hand, on any set X, we can take $\mathcal{T} = \{X, \emptyset\}$. This is called the *indiscrete topology* on X. Consider the indiscrete topological space $(X, \mathcal{T})$. We should think of this space as being very degenerate, with all of the points so close together that no matter how high of a resolution we look at X with, we won't be able to see anything other than a single clump of points.

Example 1.5. Let X be the set of real numbers $\mathbb{R}$. Let $\mathcal{T}$ be the standard topology on $\mathbb{R}$: a set $U \subseteq \mathbb{R}$ is open exactly when for any point $p \in U$, there is some real $\varepsilon > 0$ such that the open interval $(p - \varepsilon, p + \varepsilon)$ is contained in U (note that ε is allowed to vary depending on which p we choose). This topology is standard in the sense that it recovers our everyday notion of closeness when we model space with real numbers.

To see that $\mathcal{T}$ is really a topology, note first that $\mathcal{T}$ satisfies the first axiom of a topology, since $\emptyset \in \mathcal{T}$ trivially (because it contains no points) and $X \in \mathcal{T}$ by choosing $\varepsilon = 1$ (or any positive real number) for every point p.

To see that $\mathcal{T}$ satisfies the second axiom of a topology, suppose we have an arbitrary collection $\{U_\alpha\}$ of sets in $\mathcal{T}$. For any point p in $U = \bigcup_\alpha U_\alpha$, we know that $p \in U_{\alpha'}$ for some α'. Since $U_{\alpha'}$ is open, it must contain some open interval $(p - \varepsilon, p + \varepsilon)$ for some $\varepsilon > 0$. But then $(p - \varepsilon, p + \varepsilon) \subseteq U_{\alpha'} \subseteq U$, so U contains an open interval around p. Since this is true for all $p \in U$, we see that U is open.

To see that $\mathcal{T}$ satisfies the third axiom of a topology, suppose that we have two open sets U_1 and U_2. Then any point p in $U = U_1 \cap U_2$ is in particular in U_1. Since U_1 is open, there is some $\varepsilon_1 > 0$ such that $(p-\varepsilon_1, p+\varepsilon_1) \subseteq U_1$. Replacing U_1 with U_2, we see that there is some $\varepsilon_2 > 0$ such that $(p-\varepsilon_2, p+\varepsilon_2) \subseteq U_2$. Take $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Then since $(p-\varepsilon, p+\varepsilon)$ is contained in both $(p-\varepsilon_1, p+\varepsilon_1)$ and $(p-\varepsilon_2, p+\varepsilon_2)$, we see that $(p-\varepsilon, p+\varepsilon)$ is contained in both U_1 and U_2, so U contains an open interval around p. Since this is true for all $p \in U$, we see that U is open.

In algebraic geometry, we will study a topology where it is easiest to define the complements of its open sets as opposed to directly defining the open sets themselves.

Definition 1.6. If $(X, \mathcal{T})$ is a topological space, then a *closed* subset of X is a subset K such that $K = X \setminus U$ for some $U \in \mathcal{T}$.

Remark 1.7. Note that X and $\emptyset$ are always both open and closed, and in the standard topology on $\mathbb{R}$, the set $[0,1)$ is neither open nor closed. *Sets aren't doors!*

The following lemma shows that it is possible to entirely characterize a topology on a space in terms of its closed subsets.

Lemma 1.8. *Let X be a set, and let $\mathcal{C}$ be a collection of subsets of X. Then the collection*

$$\mathcal{T} = \{U \subseteq X \mid U = X \setminus K \text{ for some } K \in \mathcal{C}\}$$

is a topology on X if and only if

1. *$X \in \mathcal{C}$ and $\emptyset \in \mathcal{C}$.*
2. *If $\{K_\alpha\}$ is an arbitrary collection with $K_\alpha \in \mathcal{C}$ for all α, then $\bigcap_\alpha K_\alpha \in \mathcal{C}$.*
3. *If K_1 and K_2 are in $\mathcal{C}$, then $K_1 \cup K_2 \in \mathcal{C}$.*

If these conditions are satisfied, then $\mathcal{C}$ will be the collection of closed subsets of the topological space $(X, \mathcal{T})$.

Proof. Apply De Morgan's laws. □

Exercise 1.9. In Example 1.3, we claimed that for any set X, the collection of subsets $\mathcal{P}(X)$ satisfies the axioms of a topology. Prove this. Similarly, we claimed in Example 1.4 that the collection $\{X, \emptyset\}$ satisfies the axioms of a topology. Prove this too.

2 Rings

In general a ring is a set R, together with two binary operations, $+$ (called "addition") and $\cdot$ (called "multiplication"), a distinguished element 0, and an element $-a$ for every a in R. Letting a, b, and c be elements of R, the following axioms also have to be satisfied:

$$\begin{aligned}
&a + (b + c) = (a + b) + c \\
&a + 0 = a \\
&a + (-a) = 0 \\
&a + b = b + a \\
&a \cdot (b \cdot c) = (a \cdot b) \cdot c \\
&a \cdot (b + c) = (a \cdot b) + (a \cdot c) \\
&(b + c) \cdot a = (b \cdot a) + (b \cdot a)
\end{aligned}$$

In this seminar, we will as a rule restrict ourselves to considering commutative rings with a multiplicative identity. These rings have an additional distinguished element 1 and satisfy the following additional axioms:

$$\begin{aligned}
&a \cdot b = b \cdot a \\
&a \cdot 1 = a
\end{aligned}$$

It is almost always the case that all of the intended structure of a ring is clear just from the underlying set. In these cases, we will simply refer to the ring by its underlying set. For example, we will use $\mathbb{Z}$ to refer to the ring whose underlying set is the integers, and whose addition is integer addition, whose multiplication is integer multiplication, et cetera.

TODO: Ideals

TODO: Polynomial rings