AN INTRODUCTION TO ALGEBRAIC D-MODULES

WYATT REEVES

ABSTRACT. This paper aims to give a friendly introduction to the theory of algebraic D-modules. Emphasis is placed on examples, computations, and intuition. The paper builds up the basic theory of D-modules, concluding with a proof of the Kashiwara equivalence of categories. We assume knowledge of basic homological algebra, algebraic geometry, and sheaf theory.

CONTENTS

1. INTRODUCTION

Linearization is a wonderful thing. Let G be a Lie group acting on a smooth manifold X and let $E \to X$ be a G-equivariant vector bundle over X. By generalizing the notion of a Lie derivative, we obtain an action of $\mathfrak{g}$ on $\Gamma(X, E)$: let $s \in \Gamma(X, E)$ and $X \in \mathfrak{g}$. Define $X \cdot s \in \Gamma(X, E)$ to be the section such that

$$(X \cdot s)(p) = \frac{d}{dt} \exp(Xt) \cdot s\left(\exp(-Xt) \cdot p\right)\bigg|_{t=0}.$$

This action of $\mathfrak{g}$ on $\Gamma(X, E)$ should be thought of as the linearization of the action of G on E. The study of this linearized $\mathfrak{g}$ action has historically been quite fruitful, leading for example to the resolution of the Kazhdan-Lusztig conjectures by Beilinson and Berstein [**?**].

A group action can be thought of as a map from a group G to the automorphisms of a space X. Analogously, a Lie algebra action can be thought of as a map of $\mathfrak{g}$ into the derivations of X. The representations of $\mathfrak{g}$ are the same as modules over the universal enveloping algebra $U(\mathfrak{g})$. What is the appropriate analog to the universal enveloping algebra for the derivations of a space? We are led to study the ring of differential operators on X.

Date: DEADLINES: Draft AUGUST 19 and Final version AUGUST 31, 2019.

2. Differential Operators and D-modules

Let X be a smooth complex algebraic variety, with sheaf of regular functions $\mathcal{O}_X$. Let Θ_X be the sheaf of derivations of $\mathcal{O}_X$:

$$\Theta_X = \{\theta \in \mathrm{End}_{\mathbb{C}_X}(\mathcal{O}_X) \mid \theta(fg) = f\theta(g) + \theta(f)g\}.$$

Definition 2.1. The sheaf of differential operators on X, written D_X, is the $\mathcal{O}_X$-subalgebra of $\mathrm{End}_{\mathbb{C}_X}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and Θ_X.

Example 2.2. $X = \mathbb{A}^1$. Letting t be a coordinate for X, we know that $\Theta_X(U) = \mathcal{O}_X(U)\partial_t$. We therefore see that

$$D_X(U) = \bigoplus_{k=0}^{\infty} \mathcal{O}_X(U)\partial_t^k.$$

Example 2.3. More generally, if X is any smooth n-dimensional variety, and $p \in X$ is any point, then there is an affine open neighborhood U of p such that there are coordinate functions $x^1 \ldots x^n$ on U. In this case

$$D_X(U) = \sum_{\alpha} \mathcal{O}_X(U)\partial^{\alpha},$$

where alpha is a multi-index $\alpha = (\alpha_1 \ldots \alpha_n) \in \mathbb{N}^n$ and $\partial^{\alpha} = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$.

Example 2.4. $X = \mathbb{P}^1$. We'll first compute Θ_X. Let $U = \mathbb{P}^1 \setminus \{\infty\}$ and $V = \mathbb{P}^1 \setminus \{0\}$. Let z be a coordinate for U and let ω be a coordinate for V. On $U \cap V$ we know that $\omega = z^{-1}$. An element of Θ_X is the same data as an element of Θ_U and an element of Θ_V that glue together correctly. Since $\omega = z^{-1}$, we know that

$$\partial_\omega = \frac{dz}{d\omega}\partial_z = -\omega^{-2}\partial_z = -z^2\partial_z.$$

As a result, we see that $\Theta_X \cong \mathcal{O}(2)$ and

$$D_X \cong \bigoplus_{n=0}^{\infty} \mathcal{O}(2n).$$

If D_X is supposed to be the geometric analog of the universal enveloping algebra of a Lie algebra, then the appropriate analog of a Lie algebra representation is a module over D_X. Such objects are called D-modules. However, since D_X is not a sheaf of commutative rings, there is a difference of substance between left and right D_X-modules. We therefore make the following definitions:

Definition 2.5. A left (right) D-module on a space X is a sheaf M such that for every open subset $U \subseteq X$ we know that $M(U)$ has the structure of a left (right) $D_X(U)$-module in a way that is compatible with restriction maps.

Example 2.6. Since D_X is a subsheaf of $\mathrm{End}_{\mathbb{C}_X}(\mathcal{O}_X)$, it inherits an action on $\mathcal{O}_X$ which makes $\mathcal{O}_X$ into a left D-module.

Thankfully, D-modules are more than just a tortured analog of Lie algebra representations; they were in fact originally considered in the context of systems of linear partial differential equations. To see the connection, let $P_1 \ldots P_k$ be a collection of linear partial differential operators on X. Let $I = D_X(P_1 \ldots P_k)$ be the left ideal of D_X generated by $P_1 \ldots P_k$. Then on an open set U,

$$\begin{aligned}\mathrm{Hom}_{D_X(U)}(D_X/I(U), \mathcal{O}_X(U)) &= \{D_X(U)\text{-linear maps from } D_X(U) \text{ to } \mathcal{O}_X(U) \text{ sending } I \text{ to } 0.\} \\ &= \{f \in \mathcal{O}_X(U) \text{ such that } If = 0\} \\ &= \{f \in \mathcal{O}_X(U) \text{ such that } P_i f = 0 \text{ for } i = 1 \ldots k\}.\end{aligned}$$

We therefore see that the sheaf $\mathrm{Hom}_{D_X}(D_X/I, \mathcal{O}_X)$ is the sheaf of solutions to the system of linear PDEs given by $P_1 \dots P_k$. In this way, every system of linear PDEs $P_1 \dots P_k$ gives rise to a corresponding left D-module. Conversely, assume that M is a coherent left D-module. Then on small enough open sets U, we know that $M(U) \cong D_X^n(U)/I$, where I is a finitely generated $D_X(U)$-submodule of D_X^n. Each generator G_i for I (say there are m of them) is a collection $(P_{1i} \dots P_{ni})$ of n linear differential operators. We see that the elements of $\mathrm{Hom}_{D_X(U)}(M, \mathcal{O}_X(U))$ correspond to collections of functions $f_1 \dots f_n$, such that f_j satisfies the linear partial differential equations $P_{j1} \dots P_{jm}$. The intuition that left D-modules correspond to systems of linear PDEs will prove enlightening when we later construct the pullback and pushforward functors for D-modules.

3. Derived Categories

As it happens, techniques from homological algebra are fundamental to the study of D-modules.

Example 3.1. To get a feel for this, consider the situation where $X = \mathbb{C}$ and $P = z\partial_z - \lambda$ for $\lambda \notin \mathbb{Z}$. Let $M = D_X/D_X P$. For the purposes of this example, let $\mathcal{O}_X$ be the sheaf of analytic functions and let D_X be the sheaf of analytic differential operators. Understanding $\mathrm{Hom}_{D_X}(M, \mathcal{O}_X)$ is the same as understanding solutions of the complex differential equation $z\partial_z f - \lambda f = 0$. Any point $p \neq 0$ has a simply connected open neighborhood where the solutions are given by Cz^λ for $C \in \mathbb{C}$, so the stalk $\mathrm{Hom}_{D_X}(M, \mathcal{O}_X)_p \cong \mathbb{C}$. However, on any connected open neighborhood of 0, the only solution to the equation is 0, so $\mathrm{Hom}_{D_X}(M, \mathcal{O}_X)_0 \cong 0$.

What do the derived functors of Hom tell us about our differential equation? Note that M admits the projective resolution

$$0 \longrightarrow D_X \xrightarrow{\cdot P} D_X \longrightarrow M \longrightarrow 0$$

so the only non-vanishing derived functor of Hom is Ext^1, and

$$\mathrm{Ext}^1_{D_X}(M, \mathcal{O}_X) \cong \mathrm{coker}(P : \mathcal{O}_X \to \mathcal{O}_X),$$

where the map P just applies P to a function in $\mathcal{O}_X$. Intuitively, this cokernel is measuring how difficult it is to solve the inhomogeneous differential equation $z\partial_z f - \lambda f = g$ as g varies. At $p \neq 0$, we know that the stalk $\mathrm{Ext}^1_{D_X}(M, \mathcal{O}_X)_p \cong 0$ by an application of Morera's theorem, but the stalk at 0 is nonzero. We see that homological techniques can tell us nontrivial information about the D-modules that we're studying, and this information turns out to be important when applying D-modules to representation theory (it will be essential to consider this homological information in Theorem 6.6, for example).

Derived categories provide a technically sophisticated but highly flexible setting for doing homological algebra. The main idea is that since taking the homology groups of a chain complex is such an information-destroying operation, it should be put off as long as possible, and the primary objects of study should be the chain complexes themselves. To be able to realize this idea, we need to be able to associate to an abelian category $\mathcal{A}$ another category, $D(\mathcal{A})$, such that a left-exact functor $F : \mathcal{A} \to \mathcal{B}$ induces a functor $RF : D(\mathcal{A}) \to D(\mathcal{B})$. Moreover, the an object $A \in \mathcal{A}$ should live inside of $D(\mathcal{A})$, and $RF(\mathcal{A})$ should be a chain complex such that $H^i RF(A) = R^i F(A)$.

Evidently, the objects of $D(\mathcal{A})$ should be built in some way from chain complexes on $\mathcal{A}$. The objects of $\mathcal{A}$ will then embed as chain complexes concentrated in degree 0. If $I^\bullet$ is an injective resolution of A, then a natural choice for $RF(A)$ is $F(I^\bullet)$, the complex obtained by applying F termwise to $I^\bullet$. To avoid ambiguities with respect to choice of injective resolution, we should definitely pass to the homotopy category of chain complexes, $K(\mathcal{A})$. Moreover, it would be nice if we could say that $RF(A) = R(I^\bullet)$ because A is isomorphic to $I^\bullet$ in $D(\mathcal{A})$ and because RF acts termwise when applied to a complex of injectives. Since $I^\bullet$ is quasi-isomorphic to A, one way we could try to make A isomorphic to $I^\bullet$ is to invert the quasi-isomorphisms in $K(\mathcal{A})$. A priori, inverting quasi-isomorphisms could make our category look very strange, and it isn't clear whether a functor that applies F termwise to $I^\bullet$ would even be well-defined. However, the following lemma should give us hope:

Lemma 3.2. [2] *Let $Y^\bullet$ be a bounded-below complex of injectives. Every quasi-isomorphism $t : Y^\bullet \to Z^\bullet$ of complexes is a split injection in $K(\mathcal{A})$.*

In particular, if both $I^\bullet$ and $J^\bullet$ are complexes of injectives, then every quasi-isomorphism $t : I^\bullet \to J^\bullet$ is already an isomorphism in $K(\mathcal{A})$, so inverting quasi-isomorphisms won't affect $K(\mathcal{I})$.

Definition 3.3. Let $\mathcal{A}$ be an abelian category. Let S be the multiplicative system of quasi-isomorphisms in $K(\mathcal{A})$. Then $D(\mathcal{A}) = S^{-1} K(\mathcal{A})$.

When $\mathcal{A}$ has enough injectives, every bounded-below complex is quasi-isomorphic to a bounded-below complex of injectives, so the inclusion $K^+(\mathcal{I}) \cong D^+(\mathcal{I}) \to D^+(\mathcal{A})$ is essentially surjective, and we obtain the following result:

Theorem 3.4. [2] *Suppose $\mathcal{A}$ has enough injectives. Then $D^+(\mathcal{A})$ exists and $K^+(\mathcal{I}) \cong D^+(\mathcal{A})$. Dually, if $\mathcal{A}$ has enough projectives, then $D^-(\mathcal{A})$ exists and $K^-(\mathcal{P}) \cong D^-(\mathcal{A})$.*

Given a left-exact functor F, recall that, taken together, the collection R^iF forms a universal δ-functor extending F. Rephrasing this property in the language of derived categories gives the following definition:

Definition 3.5. Let $F : K(\mathcal{A}) \to K(\mathcal{B})$ be a morphism of triangulated categories. Write q for the functor from a homotopy category to the associated derived category. Then a right derived functor of F is a functor $RF : D(\mathcal{A}) \to D(\mathcal{B})$ and a natural transformation $\eta : qF \to RFq$ such that for any $G : D(\mathcal{A}) \to D(\mathcal{B})$ and any natural transformation $\zeta : qF \to Gq$, there exists a natural transformation $\xi : RFq \to Gq$ such that $\zeta = \xi \circ \eta$.

Remark 3.6. In other words, RF is the left Kan extension of qF along q.

In general, an additive functor $\mathcal{A} \to \mathcal{B}$ preserves chain homotopy equivalences, cones, and exact triangles, so it induces a morphism of triangulated categories $K(\mathcal{A}) \to K(\mathcal{B})$.

Example 3.7. If $F : \mathcal{A} \to \mathcal{B}$ is exact, then F preserves quasi-isomorphisms, so RF exists by the universal property of $D(\mathcal{A})$.

Theorem 3.8. [2] *Suppose that $F : K^+(\mathcal{A}) \to K(\mathcal{B})$ is a morphism of triangulated categories. Suppose that $\mathcal{A}$ has enough injectives. Then RF exists and if $I^\bullet$ is a complex all of whose entries are injective, we know that $RF(I^\bullet) \cong qF(I^\bullet)$.*

4. Filtrations and Gradings

The sheaf of differential operators carries a natural filtration:

Definition 4.1. The *order filtration* on D_X is the filtration F such that

$$F_0 D_X = \mathcal{O}_X$$
$$F_p D_X = (\Theta_X + \mathcal{O}_X) \cdot (F_{p-1} D_X)$$

Intuitively, the differential operators in F_pD_X are the ones that have order less than or equal to p. In fact, if we let X be a smooth variety and let U be an affine open neighborhood with coordinate system $x^1 \dots x^n$ (the situation of Example 2.3), then

$$F_pD_X(U) = \sum_{|\alpha|\leq p} \mathcal{O}_X(U)\partial^\alpha.$$

Since for X smooth, such U form an open cover of X, we can give an alternative characterization of F_pD_X as

$$F_pD_X(V) = \left\{ P \in D_X(V) \;\middle|\; P|_U = \sum_{|\alpha|\leq p} \mathcal{O}_X(U)\partial^\alpha \text{ for all affine coordinate charts } U \subseteq V \right\}.$$

We will call this the *local characterization* of F_pD_X.

Remark 4.2. Note that there is no coordinate-independent way of defining a grading on D_X by the order of differential operators: if we consider $X = \mathbb{A}^1 \setminus \{0\}$ and consider the coordinates z and $\omega = z^{-1}$ on X, then

$$\partial_z^2 = (-\omega^2\partial_\omega)(-\omega^2\partial_\omega) = \omega^4\partial_\omega^2 + 2\omega^3\partial_\omega,$$

so an operator that has "pure degree" in one coordinate system might not have pure degree in a different one.

Using our local characterization of F_pD_X and doing an explicit computation in local coordinates, we can obtain the following result:

Lemma 4.3. *For any $P \in F_pD_X$ and $Q \in F_qD_X$, we know that $[P, Q] = PQ - QP \in F_{p+q-1}D_X$.*

Corollary 4.4. $\operatorname{gr}^F D_X = \bigoplus_{n=0}^\infty F_nD_X/F_{n-1}D_X$ *is a sheaf of commutative algebras.*

In an affine coordinate chart U, we can be more explicit about the structure of $\operatorname{gr}^F D_X$: before passing to the associated graded algebra, $[\partial_i, x^i] = 1$, but in $\operatorname{gr}^F D_X(U)$ it is 0, so

$$\operatorname{gr}^F D_X(U) \cong \mathcal{O}_X(U) \otimes_{\mathbb{C}} \mathbb{C}[\partial_1 \dots \partial_n]$$

as an algebra.

Lemma 4.5. *Let X be a smooth variety over $\mathbb{C}$ and let $U \subseteq X$ be an affine coordinate chart. Then $\operatorname{gr}^F(U)$ is a Noetherian ring with global dimension $2\dim X$*

Proof. Since X is a variety, we know that $\mathcal{O}_X(U)$ is Noetherian, so $\operatorname{gr}^F D_X(U)$ is too by Hilbert's basis theorem. Since X is smooth, we know that $\mathcal{O}_X(U)$ has global dimension $\dim X$, so by a standard result from homological algebra (see e.g. [2]) we know that $\operatorname{gr}^F D_X(U)$ has global dimension $\dim X + \dim X$. □

For "coarse" properties, like global dimension and Noetherian-ness, the associated graded ring can give us information about our original filtered ring:

Theorem 4.6. [1] *Let (A, F) be a filtered ring such that $\operatorname{gr}^F A$ is left (right) Noetherian. Then A is left (right) Noetherian. Moreover, the left (right) global dimension of A is bounded above by that of $\operatorname{gr}^F A$.*

So we see that D_X is locally Noetherian with finite global dimension. We also have the following two results:

Theorem 4.7. [1] *Let M be a quasicoherent D_X module. Then M embeds into a quasicoherent D_X module I which is injective in* $\mathrm{Mod}_{qc}(D_X)$.

Theorem 4.8. [1] *Suppose that X is a quasi-projective variety. Let M be a quasicoherent D_X module. Then M is a quotient of a quasicoherent D_X module F which is locally free.*

In order to have Theorem 4.8, from now on we will only work with quasi-projective varieties. Taken together, Theorem 4.6, Theorem 4.7, and Theorem 4.8 imply that we can work in the category $D^b_{qc}(D_X)$ of complexes of sheaves which are bounded and have quasicoherent cohomology sheaves. In particular, we have the result

Theorem 4.9. [1] *Every object of $D^b_{qc}(D_X)$ is represented by a bounded complex of locally projective quasicoherent D_X modules.*

5. Pullback of D-modules

Given a regular morphism f between smooth complex algebraic varieties X and Y, we will now construct two functors obtained from f that relate D_X modules and D_Y modules. The first of these functors is the pullback functor, f^*. Intuitively, pullback takes a D_Y module, finds its sheaf of solutions $\mathcal{F}$, pulls those back to X, and then gives the D_X module whose solutions are $f^*\mathcal{F}$. To preserve higher homological information, we work in the derived setting.

Definition 5.1. If $f : X \to Y$ is a regular morphism of smooth complex algebraic varieties, then the pullback of D-modules along f is the functor $f^! : D^b(Y) \to D^b(X)$ given by

$$f^!M = (\mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} f^{-1} M)[\dim X - \dim Y].$$

The D_X module structure is given by

$$g \cdot (f \otimes m) = gf \otimes m$$

for $g \in \mathcal{O}_X$ and

$$\theta \cdot (f \otimes m) = \theta(f) \otimes m + f \otimes df(\theta) \cdot m$$

for $\theta \in \Theta_X$.

Remark 5.2. The shifting of degree will make the statements of some key theorems, like Theorem 6.10 and Theorem 6.11, more natural.

Example 5.3. Open embeddings. Let U be an open subset of X and let j be the inclusion. Then $\dim U = \dim X$. Moreover, $j^{-1}\mathcal{O}_X = \mathcal{O}_U$ and $j^{-1} M = M|_U$. Since $\mathcal{O}_U \otimes_{j^{-1}\mathcal{O}_X} j^{-1} M = M|_U$, and since restriction of a sheaf to an open subset is an exact functor, we know that

$$j^!M = (\mathcal{O}_U \otimes^L_{j^{-1}\mathcal{O}_X} j^{-1} M)[\dim X - \dim U] = \mathcal{O}_U \otimes_{j^{-1}\mathcal{O}_X} j^{-1} M = M|_U$$

More generally, we can say that

Theorem 5.4. *Let $f : X \to Y$ be a flat morphism of smooth algebraic varieties and let M be a D_Y-module. Then $H^k(f^!M) = 0$ unless $k = \dim X - \dim Y$.*

Proof. Because f is flat, we know that the functor $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}-$ is exact, so $\mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} f^{-1}- \cong \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}-$, and therefore $f^!M \cong (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1} M)[\dim X - \dim Y]$ □

Example 5.5. Closed embeddings. Let $Y = \mathbb{A}^2$, with regular functions $\mathcal{O}_Y = \mathbb{C}[x, y]$, and let X be the smooth subvariety cut out by the equation $y = 0$. Let i denote the closed embedding $X \to Y$. First consider $M = D_Y/D_Y \cdot y$. In the category $D^b(f^{-1} D_Y\text{-Mod})$, we know

$$M \cong\ 0 \longrightarrow f^{-1} D_Y \xrightarrow{\cdot y} f^{-1} D_Y \longrightarrow 0\ ,$$

which is a complex of free (and therefore projective) $f^{-1} \mathcal{O}_Y$ modules. As a result, we can apply $\otimes_{f^{-1} \mathcal{O}_Y}$ termwise to obtain:

$$\begin{aligned} \mathcal{O}_X \otimes^L_{f^{-1} \mathcal{O}_Y} M &\cong\ 0 \longrightarrow \mathcal{O}_X \otimes_{f^{-1} O_Y} f^{-1} D_Y \xrightarrow{\cdot y} \mathcal{O}_X \otimes_{f^{-1} O_Y} f^{-1} D_Y \longrightarrow 0 \\ &\cong\ 0 \longrightarrow D_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_y] \xrightarrow{\cdot y} D_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_y] \longrightarrow 0 \end{aligned}$$

Note that y acts on ∂_y^n on the right by $\partial_y^n y = y\partial_y^n + n\partial_y^{n-1}$. Since y acts on $\mathbb{C}[\partial_y]$ on the left by 0, we see that acting on a polynomial $p(\partial_y)$ the right by y just differentiates p. As such, we see that $H^0(\mathcal{O}_X \otimes^L_{f^{-1} \mathcal{O}_Y} M) \cong 0$ and $H^{-1}(\mathcal{O}_X \otimes^L_{f^{-1} \mathcal{O}_y} M) \cong D_X$ (and the other cohomology groups all vanish). The cohomology groups of $f^! M$ are just shifted by $\dim X - \dim Y = -1$.

This result makes good intuitive sense. The module $D_Y/D_Y \cdot y$ corresponds to the differential equation $yf = 0$, which only has the solution $f = 0$. This function pulls back to 0, so the degree zero part of the pullback complex should be small (it should have few solutions). On the other hand, any solution g to the inhomogeneous equation $yf = g$ will pull back to 0, so we would expect the degree 1 part of the chain complex to be large.

In general, we might not have such a nice projective resolution of M, so to analyze pullbacks along an arbitrary closed embedding of smooth varieties, we choose instead to resolve $\mathcal{O}_X$ as a $f^{-1} \mathcal{O}_Y$ module.

Theorem 5.6. [2] *Let X be a smooth closed m-dimensional subvariety of the smooth n-dimensional algebraic variety Y. Let $i : X \to Y$ be the embedding. Let U be an affine coordinate chart of Y with local coordinates $y^1 \dots y^n$ such that $y^1 \dots y^m$ form a coordinate system for $X \cap U$. Then*

$$0 \longrightarrow K_{n-m} \longrightarrow \dots \longrightarrow K_0 \longrightarrow \mathcal{O}_X(U \cap X) \longrightarrow 0$$

gives a free resolution of $\mathcal{O}_X(U \cap X)$ as a $i^{-1} \mathcal{O}_Y(U \cap X)$-module, where

$$K_j = \bigwedge^j \left(\bigoplus_{k=m+1}^n i^{-1} \mathcal{O}_Y(U \cap X) dy^k \right)$$

and the differential $d : K_j \to K_{j-1}$ is given by

$$d(f dy^{k_1} \wedge \ldots \wedge dy^{k_j}) \mapsto \sum_{l=1}^j y_{k_l} f dy^{k_1} \wedge \ldots \wedge \widehat{dy^{k_l}} \wedge \ldots \wedge dy^{k_j}$$

and the map $K_0 = f^{-1} \mathcal{O}_Y(U \cap X) \to \mathcal{O}_X(U \cap X)$ is the pullback of functions. Moreover, each K_j can be patched together into a locally free sheaf of $f^{-1} \mathcal{O}_Y$ modules in such a way that we obtain a resolution

$$0 \longrightarrow K_{n-m} \longrightarrow \dots \longrightarrow K_0 \longrightarrow \mathcal{O}_X \longrightarrow 0$$

at the level of sheaves.

Corollary 5.7. *Let X be a smooth closed m-dimensional subvariety of the smooth n-dimensional algebraic variety Y. Let $i : X \to Y$ be the embedding and let M be a D_Y-module. Then $H^k(i^! M) = 0$ unless $0 \leq k \leq n - m$.*

Proof. We can resolve $\mathcal{O}_X$ by the Koszul resolution, so that in $D^b(D_X)$

$$\mathcal{O}_X \otimes^L_{i^{-1}\mathcal{O}_Y} M \cong 0 \longrightarrow K_{n-m} \otimes_{f^{-1}\mathcal{O}_Y} M \longrightarrow \dots \longrightarrow K_0 \otimes_{f^{-1}\mathcal{O}_Y} M \longrightarrow 0\ ,$$

which can only have nonzero cohomology in degrees between $m-n$ and 0. Since $i^! M = (\mathcal{O}_X \otimes^L_{i^{-1}\mathcal{O}_Y} M)[m-n]$, we obtain the result after shifting degrees. □

We'll now introduce a $(D_X, f^{-1} D_Y)$-bimodule that gives a convenient way of packaging together the data involved in transferring a D_Y module to a D_X module:

Definition 5.8. If $f : X \to Y$ is a regular morphism of smooth complex algebraic varieties, then the *transfer bimodule* is the $(D_X, f^{-1} D_Y)$-bimodule

$$D_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1} D_Y,$$

where $\mathcal{O}_X$ and Θ_X act like in Definition 5.1

Example 5.9. Closed embedding. Let X be a smooth closed m-dimensional subvariety of the smooth n-dimensional algebraic variety Y. Let i be the embedding map. Then in some affine neighborhood U around any point p of X we can find coordinates $y^1 \dots y^n$ such that $y^1 \dots y^m$ are a coordinate system for $X \cap U$. In these coordinates

$$D_{X\to Y} \cong D_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_{m+1} \dots \partial_n],$$

giving us a local description of $D_{X\to Y}$.

We can rephrase the pullback of D-modules in terms of the transfer bimodule:

Lemma 5.10. *If $f : X \to Y$ is a regular morphism of smooth complex algebraic varieties, then*

$$f^! M = (D_{X\to Y} \otimes^L_{f^{-1} D_Y} M)[\dim X - \dim Y]$$

Proof. Since by D_Y is locally free as an $\mathcal{O}_Y$ module, we see that

$$\begin{aligned} D_{X\to Y} \otimes^L_{f^{-1} D_Y} M &= (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1} D_Y) \otimes^L_{f^{-1} D_Y} M \\ &= (\mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} f^{-1} D_Y) \otimes^L_{f^{-1} D_Y} M \\ &= \mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} M, \end{aligned}$$

□

To finish our discussion of pulling back D-modules, let's note the following essential property of pullbacks:

Theorem 5.11. *If $f : X \to Y$ and $g : Y \to Z$ are maps of smooth varieties X, Y, and Z, then*

$$(g \circ f)^! = f^! \circ g^!$$

Proof. Let $M \in D^b(D_Z)$. Then

$$\begin{aligned} f^! \circ g^! M &= \mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} f^{-1}(\mathcal{O}_Y \otimes^L_{g^{-1}\mathcal{O}_Z} g^{-1} M) \\ &= \mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{O}_Y \otimes^L_{f^{-1} g^{-1}\mathcal{O}_Z} f^{-1} g^{-1} M \\ &= \mathcal{O}_X \otimes^L_{(g\circ f)^{-1}\mathcal{O}_Z} (g \circ f)^{-1} M \\ &= (g \circ f)^! M \end{aligned}$$

□

We can also rephrase this fact in terms of the transfer bimodule as follows:

Theorem 5.12. *If $f : X \to Y$ and $g : Y \to Z$ are maps of smooth varieties X, Y, and Z, then*

$$D_{X\to Y} \otimes^L_{f^{-1} D_Y} f^{-1} D_{Y\to Z} \cong D_{X\to Z}$$

Proof. This is another more-or-less formal manipulation, and all of the manipulations are contained within Lemma 5.10 and Theorem 5.11. □

6. Pushforward of D-modules

We'll now consider the pushforward functor of D-modules, f_*. The fundamental difficulty in defining the pushforward functor is that functions don't generally push forward along a map of algebraic varieties. As a result, if we want to make sense of a pushforward of D-modules, we need to reinterpret our D-modules in terms of some other kind of object that we *can* push forward: distributions give us exactly what we need.

Informally, a distribution is an object that you can integrate functions against. Because of this, to specify a distribution, it suffices to describe how any function integrates against it. As a result, if $\phi : X \to Y$ is a map between smooth complex algebraic varieties and δ is a distribution on X, then we can define $\phi_*\delta$ to be the distribution on Y such that

$$\int_Y \phi_*\delta f = \int_X \delta\phi^* f$$

for all regular f on Y. Although until now we have been able to avoid talking about right D-modules, it turns out that the most natural action of D_X on distributions is a right action. If $P \in D_X$, then we can define an action of P on δ by

$$\int_X (P \cdot \delta) f = \int_X \delta(P \cdot f),$$

and this forces our hand: if $P, Q \in D_X$, then we have

$$\int_X (Q \cdot (P \cdot \delta)) f = \int_X (P \cdot \delta)(Q \cdot f) = \int_X \delta(P \cdot (Q \cdot f)) = \int_X \delta(PQ \cdot f) = \int_X (PQ \cdot \delta) f.$$

Because of this, from now on we will write $\delta \cdot P$ for the action of a differential operator on a distribution. In the algebraic setting, the correct notion of a distribution on X is a section of the sheaf of top-dimensional differential forms Ω_X on X. In this setting, we can write down the right action of D_X on Ω_X more explicitly. First, suppose that ξ is a vector field on X. Let ω be a section of Ω_X. Then

$$\begin{aligned}\int_X \omega(\xi f) &= \int_X \omega \wedge \iota_\xi df \\ &= (-1)^{\dim X+1} \int_X \iota_\xi\omega \wedge df \\ &= -\int_X (d\iota_\xi\omega) f \\ &= \int_X (-\mathcal{L}_\xi\omega) f,\end{aligned}$$

so if $\theta \in \Theta_X$, then $\omega \cdot \theta = -\operatorname{Lie}_\theta \omega$. If $f \in \mathcal{O}_X$, then $\omega \cdot f = f\omega$. To be even more explicit let U be an affine neighborhood with coordinate functions $\{x^1 \dots x^n\}$. Then we can trivialize Ω_X on U via $\Omega_X(U) \cong \mathcal{O}_X(U) dx^1 \wedge \dots \wedge dx^n$ and write

$$(f(x^1 \dots x^n) dx^1 \wedge \dots \wedge dx^n) \cdot P = (P^* f(x^1 \dots x^n)) dx^1 \wedge \dots \wedge dx^n,$$

where P^* is the formal adjoint of P: for $P = \sum_\alpha f(x^1 \dots x^n) \partial_\alpha$ in D_X, we define

$$P^* = \sum_\alpha (-\partial)_\alpha f(x^1 \dots x^n)$$

in D_X.

Since we see that pushforwards are most natural in the context of distributions, not functions, and that distributions are most naturally understood in terms of right D-modules, we will first define the pushforward of right D-modules:

Definition 6.1. If $f : X \to Y$ is a regular morphism of smooth complex algebraic varieties, then the pushforward of right D-modules along f is the functor $f_* : D^b(X) \to D^b(Y)$ given by

$$f_* M = Rf_*(M \otimes^L_{D_X} D_{X \to Y})$$

Example 6.2. Closed embeddings. Let X be a smooth closed m-dimensional subvariety of the smooth n-dimensional algebraic variety Y. Let i be the embedding. We have seen already in Example 5.9 that $D_{X \to Y}$ is locally free over D_X, so $M \otimes_{D_X} D_{X \to Y}$ is exact. Furthermore, since a closed embedding is in particular an affine morphism, we know that the sheaf pushforward i_* is also exact, so the D-module pushforward i_* is exact, and

$$i_* M = i_*(M \otimes_{D_X} D_{X \to Y}).$$

On an affine coordinate chart U with coordinates $y^1 \dots y^n$ such that $y^1 \dots y^m$ are coordinates for $X \cap U$, we know that

$$i_* M(U) = (M \otimes_{D_X} (D_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_{m+1} \dots \partial_n]))(U) \cong M(U) \otimes_{\mathbb{C}} \mathbb{C}[\partial_{m+1} \dots \partial_n].$$

Intuitively, this process "infinitesimally thickens" M to all orders in the directions normal to X. At the level of distributions, the pushforward of a distribution on X should be a distribution on Y that's supported on X. These distributions will all satisfy the distributional equation $y^i \delta = 0$ for $m + 1 \leq i \leq n$, so $m \otimes y^i$ should be zero in $i_* M(U)$, as is the case.

Example 6.3. Open embeddings. For an open embedding $j : X \to Y$ we know $D_{X \to Y} \cong D_X$, so $M \otimes D_{X \to Y}$ is exact and

$$j_* M \cong Rj_*(M).$$

Example 6.4. Projections. Let $X = \mathbb{P}^1$, let Y be a point, and let $M = \Omega_X$. Since Y is a point, we know that $\mathcal{O}_Y = D_Y$, so $D_{X \to Y} \cong \mathcal{O}_X$. In $D^b(X)$, we know that

$$\mathcal{O}_X \cong \; 0 \longrightarrow D_X \cdot \Theta_X \longrightarrow D_X \longrightarrow 0$$

where the right hand side is a complex of locally free D_X modules, so

$$M \otimes^L_{D_X} \mathcal{O}_X \cong \; 0 \longrightarrow M \otimes_{D_X} D_X \cdot \Theta_X \longrightarrow M \otimes_{D_X} D_X \longrightarrow 0$$

$$\cong \; 0 \longrightarrow \Omega_X \otimes_{\mathcal{O}_X} \Theta_X \xrightarrow{\phi} \Omega_X \longrightarrow 0 \,,$$

where in local coordinates ϕ maps $f dx \otimes \partial_x \mapsto (-\partial_x f)dx$. Note in particular that this acts by 0 on global sections. We know that $\Omega_X \otimes_{\mathcal{O}_X} \Theta_X \cong \mathcal{O}(0)$ and $\Omega_X \cong \mathcal{O}(-2)$. As a result,

$$\begin{aligned} M \otimes^L_{D_X} \mathcal{O}_X &\cong 0 \longrightarrow \mathcal{O}(0) \longrightarrow \mathcal{O}(-2) \longrightarrow 0 \\ &\cong 0 \longrightarrow \mathcal{O}(0) \longrightarrow \mathcal{O}(0) \oplus \mathcal{O}(0) \longrightarrow \mathcal{O}(2) \longrightarrow 0 \end{aligned}$$

gives a resolution of $M \otimes^L_{D_X} \mathcal{O}_X$ by Γ-acyclic sheaves, so

$$\begin{aligned} f_* M &\cong 0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^2 \longrightarrow \mathbb{C}^3 \longrightarrow 0 \\ &\cong 0 \longrightarrow \mathbb{C} \longrightarrow 0 \longrightarrow \mathbb{C} \longrightarrow 0\,. \end{aligned}$$

We are now in a position to get an idea of why all of this homological nonsense is important to keep track of. Let $f^N_* : D_X - ^R\text{mod} \to D_X - ^R\text{mod}$ be the "naive pushforward" of right D-modules:

$$f^N_* M = f_*(M \otimes_{D_X} D_{X\to Y})$$

Example 6.5. The naive pushforward doesn't compose correctly.

TODO: Give example where non-derived version fails to compose correctly

On the other hand, the derived pushforward does compose correctly:

Theorem 6.6. *If $f : X \to Y$ and $g : Y \to Z$ are maps of smooth varieties X, Y, and Z, then*

$$(g \circ f)_* = g_* \circ f_*$$

Proof. First note that by Theorem 4.8 we can replace $D_{Y\to Z}$ with a bounded-above complex $F^\bullet$ of locally free sheaves in $D^-_{qc}(D_Y)$. We will show locally that

$$Rf_*(M \otimes^L_{D_X} D_{X\to Y}) \otimes^L_{D_Y} F^\bullet \cong Rf_*(M \otimes^L_{D_X} D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1} F^\bullet)$$

in a natural way, so that we obtain a global isomorphism. Let $F_j = D_U^{I_j}$ for some index set I_j and some affine U where F_j restricts to a free D_U-module. Then because Rf_* naturally commutes with direct sums, we know term-by-term that

$$\begin{aligned} Rf_*(M \otimes^L_{D_X} D_{X\to Y}) \otimes_{D_Y} F_j &\cong Rf_*(M \otimes^L_{D_X} D_{X\to Y})^{\oplus I_j} \\ &\cong Rf_*\left((M \otimes^L_{D_X} D_{X\to Y})^{\oplus I_j}\right) \\ &\cong Rf_*(M \otimes^L_{D_X} D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1} F_j). \end{aligned}$$

By naturality, this gives an isomorphism of complexes that globalizes. Since $D_{Y\to Z} \cong F^\bullet$ in $D^-_{qc}(D_Y)$, this implies that

$$Rf_*(M \otimes^L_{D_X} D_{X\to Y}) \otimes^L_{D_Y} D_{Y\to Z} \cong Rf_*(M \otimes^L_{D_X} D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1} D_{Y\to Z}).$$

As a result,

$$\begin{aligned} g_* \circ f_* M &\cong Rg_*(Rf_*(M \otimes^L_{D_X} D_{X\to Y}) \otimes^L_{D_Y} D_{Y\to Z}) \\ &\cong Rg_*(Rf_*(M \otimes^L_{D_X} D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1} D_{Y\to Z})) \\ &\cong R(g\circ f)_*(M \otimes^L_{D_X} D_{X\to Z}) \\ &\cong (g \circ f)_* M, \end{aligned}$$

where we have used Theorem 5.12 in obtaining the third isomorphism. □

In what follows, we would like to talk about the relationship between pushing forward and pulling back D-modules. However, since we defined the D-module pullback for left D-modules and the D-module pushforward for right D-modules, we need to find some way to turn left D-modules into right D-modules and vice versa.

Definition 6.7. Given a left D_X-module M, we can give the tensor product

$$\Omega_X \otimes_{\mathcal{O}_X} M$$

the structure of a right D_X module by stipulating that

$$(\omega \otimes m) \cdot f = f\omega \otimes m$$

for $f \in \mathcal{O}_X$ and

$$(\omega \otimes m) \cdot \theta = (\omega \cdot \theta) \otimes m - \omega \otimes (\theta \cdot m)$$

for $\theta \in \Theta_X$, where $\omega \cdot \theta = \mathrm{Lie}_\theta\, \omega$.

Definition 6.8. Given a right D_X-module M, we can give the tensor product

$$\Omega_X^{\otimes -1} \otimes_{\mathcal{O}_X} M \cong \mathrm{Hom}_{\mathcal{O}_X}(\Omega_X, M)$$

the structure of a left D_X module by stipulating that

$$(f \cdot \phi)(\omega) = f \cdot \phi(\omega)$$

for $f \in \mathcal{O}_X$ and

$$(\theta \cdot \phi)(\omega) = -\phi(\omega) \cdot \theta + \phi(\omega \cdot \theta)$$

for $\theta \in \Theta_X$

The functors $\Omega_X \otimes_{\mathcal{O}_X} \cdot : \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_X^{op})$ and $\Omega_X^{\otimes -1} \otimes_{\mathcal{O}_X} \cdot : \mathrm{Mod}(D_X^{op}) \to \mathrm{Mod}(D_X)$ are called the *side changing* functors.

Lemma 6.9. $\Omega_X \otimes_{\mathcal{O}_X} \cdot : \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_X^{op})$ *is an equivalence of categories with inverse* $\Omega_X^{\otimes -1} \otimes_{\mathcal{O}_X} \cdot : \mathrm{Mod}(D_X^{op}) \to \mathrm{Mod}(D_X)$.

Proof. Check that the natural isomorphisms of $\mathcal{O}_X$ modules are D_X equivariant with the actions defined above. ◻

Theorem 6.10. *Let X be a smooth closed m-dimensional subvariety of the smooth n-dimensional algebraic variety Y. Let i be the embedding. Then $i_* : D^b_{qc}(D_X) \to D^b_{qc}(D_Y)$ is left adjoint to $i^! : D^b_{qc}(D_Y) \to D^b_{qc}(D_X)$.*

Proof. In this proof we will work with right D-modules. We will show on an open cover by affines U that $\mathrm{Hom}_{D^b_{qc}(D_U)}(i_*M, N)$ is naturally isomorphic to $\mathrm{Hom}_{D^b_{qc}(D_{U\cap X})}(M, i^!N)$. By the naturality of the isomorphism, these maps will then glue together to give an isomorphism globally. Around each point p of X, let U be an affine coordinate chart with coordinates $y^1 \dots y^n$ such that $y^1 \dots y^m$ are coordinates for $X \cap U$. Because we're working with quasicoherent sheaves on an affine space, we'll just write M for $M(U)$ and so on. Then we have

$$\begin{aligned}\mathrm{Hom}_{D_U}(i_*M, N) &= \mathrm{Hom}_{D^b_{qc}(D_U)}(M \otimes^L_{D_{U\cap X}} D_{(U\cap X)\to U}, N)\\ &\cong \mathrm{Hom}_{D^b_{qc}(D_{U\cap X})}(M, R\,\mathrm{Hom}_{D_U}(D_{(U\cap X)\to U}, N)),\end{aligned}$$

so it suffices to show that $R\operatorname{Hom}_{D_U}(D_{(U\cap X)\to U}, N) \cong i^! N$. To see this, note that

$$\begin{aligned} R\operatorname{Hom}_{D_U}(D_{(U\cap X)\to U}, N) &= R\operatorname{Hom}_{D_U}(\mathcal{O}_{U\cap X} \otimes_{\mathcal{O}_U} D_U, N) \\ &\cong R\operatorname{Hom}_{\mathcal{O}_U}(\mathcal{O}_{U\cap X}, N) \\ &\cong N \otimes^L_{\mathcal{O}_U} R\operatorname{Hom}_{\mathcal{O}_U}(\mathcal{O}_{U\cap X}, \mathcal{O}_U) \end{aligned}$$

Because we are studying a closed embedding, we can resolve $\mathcal{O}_{U\cap X}$ with the Koszul resolution to see that

$$R\operatorname{Hom}_{\mathcal{O}_U}(\mathcal{O}_{U\cap X}, \mathcal{O}_U) \cong \ 0 \longrightarrow K_0^* \longrightarrow \dots \longrightarrow K_{n-m}^* \longrightarrow 0 \ ,$$

and since there is a canonical non-degenerate bilinear pairing $K_j \otimes_{\mathcal{O}_U} K_{n-m-j} \to K_{n-m}$ we see that

$$\begin{aligned} R\operatorname{Hom}_{\mathcal{O}_U}(\mathcal{O}_{U\cap X}, \mathcal{O}_U) &\cong \Big(0 \longrightarrow K_{n-m} \longrightarrow \dots \longrightarrow K_0 \longrightarrow 0 \Big) \otimes_{\mathcal{O}_U} K_{n-m}^* \\ &\cong \mathcal{O}_{U\cap X}[m-n] \otimes_{\mathcal{O}_U} K_{n-m}^* \\ &\cong \Omega_U^{\otimes -1} \otimes_{\mathcal{O}_U} \Omega_{U\cap X}[m-n]. \end{aligned}$$

As a result,

$$\begin{aligned} R\operatorname{Hom}_{D_U}(D_{(U\cap X)\to U}, N) &\cong N \otimes^L_{\mathcal{O}_U} R\operatorname{Hom}_{\mathcal{O}_U}(\mathcal{O}_{U\cap X}, \mathcal{O}_U) \\ &\cong N \otimes^L_{\mathcal{O}_U} \Omega_U^{\otimes -1} \otimes_{\mathcal{O}_U} \Omega_{U\cap X}[m-n] \\ &\cong N \otimes^L_{D_U} D_{U\leftarrow(U\cap X)}[m-n] \\ &\cong i^! N. \end{aligned}$$

Because of the naturality of all of the isomorphisms used in this argument, we see that $\operatorname{Hom}_{D_U}(i_* M, N)$ is naturally isomorphic to $\operatorname{Hom}_{D(U\cap X)}(M, i^! N)$. □

A celebrated result of Kashiwara gives additional information about the pushforward in the case of a closed embedding.

Theorem 6.11. *Let X be a smooth closed m-dimensional subvariety of the smooth n-dimensional algebraic variety Y. Let i be the embedding. Then $i_* : \operatorname{Mod}_{qc}(D_X) \to \operatorname{Mod}^X_{qc}(D_Y)$ induces an equivalence of categories, with inverse $i^!$.*

Remark 6.12. Implicit in this statement is the fact that $i^! N$ only has nontrivial cohomology in degree 0 when $N \in \operatorname{Mod}^X_{qc}(D_Y)$.

Proof. In this proof we will work with left D-modules. First note that by our explicit computation (Example 6.2) we know that $i_* M \in \operatorname{Mod}^X_{qc}(D_Y)$. Now need to check that $i^! N \in \operatorname{Mod}_{qc}(D_X)$ when $N \in \operatorname{Mod}^X_{qc}(D_Y)$ and that the maps $M \to i^! i_* M$ and $i_* i^! N \to N$ from the adjunction (Theorem 6.10) are isomorphisms. All of these facts can be checked affine-locally. Furthermore, because of the compositionality of $i^!$ and i_* (Theorem 5.11 and Theorem 6.6) we can induct on the codimension of X in Y to reduce to the codimension 1 case. We have therefore reduced the problem to studying an affine coordinate chart U with coordinate functions $y^1 \dots y^n$ such that $y^1 \dots y^{n-1}$ are coordinates for $U \cap X$. Write $y = y^n$ for the defining equation of $X \cap U$ in U.

Since we are working with quasicoherent sheaves on an affine chart, we will abuse notation and identify sheaves with their global sections. Since $\mathcal{O}_{U\cap X} \cong \mathcal{O}_U / y\mathcal{O}_U$, we know that

$$\mathcal{O}_{U\cap X} \otimes^L_{\mathcal{O}_U} N \cong \ 0 \longrightarrow N \xrightarrow{\ y\cdot\ } N \longrightarrow 0$$

So $H^0(i^! N) \cong \ker(y\cdot : N \to N)$ and $H^1(i^! N) \cong \operatorname{coker}(y\cdot : N \to N)$. Consider the Euler operator $E = y\partial_y$, and let N_λ the eigenspace of E acting on N with eigenvalue λ. We will first show the following key characterization of N in terms of E:

$$\operatorname{Ann}(y^k) = \bigoplus_{i=-k}^{-1} N_i.$$

To see that $N_i \subseteq \operatorname{Ann}(y^i)$, consider some $n \in N_i$. Since N is supported on X, there is some k such that $y^k n = 0$. Let k' be the smallest such k. Note that

$$0 = \partial_y y^{k'} n = k' y^{k'-1} n - y^{k'-1} E n = (k' - i) y^{k'-1} n,$$

so if $k' > i$ then $k' - 1$ also annihilates n and we obtain a contradiction. Now we'll show by induction that

$$\operatorname{Ann}(y^k) \subseteq \bigoplus_{i=-k}^{-1} N_i.$$

When $k = 1$, note that if $n \in \operatorname{Ann}(y)$, then

$$0 = \partial_y y n = En + n,$$

so $n \in N_{-1}$. Now assuming the result for $k-1$, note that if $n \in \operatorname{Ann}(y^k)$, then

$$y^{k-1}(En + kn) = y^k \partial_y n + (\partial_y y^k - y^k \partial_y) n = \partial_y y^k n = 0,$$

so

$$En + kn = \sum_{i=-1}^{-k+1} n_i = \sum_{i=-1}^{-k+1} (k+i) n_i',$$

where $n_i' = n_i/(k+i)$ is in N_i. Then

$$\begin{aligned} E\left(n - \sum_{i=-1}^{-k+1} n_i'\right) &= En - \sum_{i=-1}^{-k+1} i n_i' \\ &= -kn + \sum_{i=-1}^{-k+1} (k+i) n_i' - \sum_{i=-1}^{-k+1} i n_i' \\ &= -k\left(n - \sum_{i=-1}^{-k+1} n_i'\right), \end{aligned}$$

so we can write n as a sum of elements in N_i for $-k \leq i \leq -1$. Since this is true for all $n \in \operatorname{Ann}(y^k)$, we obtain the desired inclusion. Since N is supported in X, every element of N is in $\operatorname{Ann}(y^k)$ for some large enough k, so

$$N = \bigoplus_{i=-\infty}^{-1} N_i.$$

Note that $Ey = y(E+1)$ and $E\partial_y = \partial_y(E-1)$, so $y\cdot$ maps N_i to N_{i+1} and $\partial_y\cdot$ maps N_i to N_{i-1}. Since $E = y\partial_y$ is invertible on each N_i (for $i \leq -1$), we see that ∂_y maps N_i isomorphically to N_{i-1}. On the other hand, $\partial_y y$ is invertible on N_i for $i \leq -2$ and zero when $i = -1$. This shows that $H^0(i^! N) \cong N_{-1}$, that $H^1(i^! N) \cong 0$, and that $N \cong \mathbb{C}[\partial_y] \otimes_{\mathbb{C}} N_{-1}$. From our explicit computation of the pushforward for a closed embedding (Example 6.2), we see that $M \cong i^! i_* M$ for any $M \in \operatorname{Mod}_{qc}(D_X)$ and that $i_* i^! N \cong N$ for any $N \in \operatorname{Mod}_{qc}^X(D_Y)$. □

Note that this is not at all true if D_X is replaced by $\mathcal{O}_X$.

Example 6.13. Let $Y = \mathbb{A}^1$ and X be the origin. Then the $\mathcal{O}_Y$ module $\mathcal{O}_Y/y^2$ is supported on X but isn't the pushforward of any $\mathcal{O}_X$ module, since y acts by 0 on any such pushforward but doesn't act by 0 on $\mathcal{O}_Y/y^2$.

Intuitively, we can have $\mathcal{O}_Y$ with higher-order information supported at X that won't be detected by the $\mathcal{O}_X$-module pushforward. However, for D-modules, because of the D_Y action on the pushforward, the pushed-forward module is "infinitesimally thickened" and can sniff out everything that's supported on X.

Acknowledgments

I would like to thank my mentor, Santiago Chaves Aguilar, for his guidance this summer. I would like to thank Sasha Beilinson for the lovely conversations we had. I would like to thank Sam Raskin for helping me understand D-module pushforwards, and I would like to thank Anshul Adve for our conversations on complex ODEs. I would like to thank the friends I met this summer for making the REU generally excellent – if you've skipped straight to the acknowledgements, you're surely included. Finally, I would very much like to thank Peter May for all of the time and effort that he has dedicated to running an amazing REU.

References

[1] Riyoshi Hotta, Kiyoshi Takeuchi, Toshiyuki Tanisaki. D-modules, Perverse Sheaves, and Representation Theory. Translated by Kiyoshi Takeuchi. Birkhauser Boston. 2008.
[2] Charles Weibel. An Introduction to Homological Algebra. Cambridge University Press. 1994.