NICK'S THESIS

1. Deformation to the normal cone in formal geometry.

1.1. **Construction of the deformation.** Let $f : \mathcal{X} \to \mathcal{Y}$ be inf-schematic (technical condition ensuring that IndCoh-pushforward exists) nil-isomorphism of laft-def prestacks. Recall that we have $T^*(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{fake}_{laft}$.

By Serre duality and convergence of $\mathrm{IndCoh}(\mathcal{X})$ we get

$$(\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{fake}_{laft})^{op} \cong \mathrm{IndCoh}(\mathcal{X}). \tag{1.1}$$

Definition 1.1. The object corresponding to $T^*(\mathcal{X})$ under the equivalence (1.1) is the tangent complex $T(\mathcal{X})$.

We will denote by $T(\mathcal{X}/\mathcal{Y})$ the fiber of the map $T(\mathcal{X}) \to f^! T(\mathcal{Y})$. We will call $T(\mathcal{X}/\mathcal{Y})[1]$ the normal bundle to $\mathcal{X}$ in $\mathcal{Y}$.

We also introduce the following analogs of total spaces of tangent bundles in formal geometry:

Definition 1.2.
1. $\mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X})) := \mathrm{Maps}(k[\epsilon]/\epsilon^2, \mathcal{X})^{\wedge}_{\mathcal{X}}$,
2. $\mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y})) := (\mathrm{Maps}(k[\epsilon]/\epsilon^2, \mathcal{X}) \times_{\mathrm{Maps}(k[\epsilon]/\epsilon^2, \mathcal{Y})} \mathcal{Y})^{\wedge}_{\mathcal{X}}$,
3. $\mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y})[1]) := (\mathrm{Maps}(k[a]/a^2, \mathcal{X}) \times_{\mathrm{Maps}(k[a]/a^2, \mathcal{Y})} \mathcal{Y})^{\wedge}_{\mathcal{X}}$, where a has homological degree 1.

The goal of this subsection is to construct $\mathcal{Y}_{scaled} \in \mathrm{PreSt}^{laft-def}_{\mathcal{X} \times \mathbb{A}^1 // \mathcal{Y} \times \mathbb{A}^1}$, such that

1. $\mathcal{Y}_{scaled} \to \mathcal{Y} \times \mathbb{A}^1$ is inf-schematic nil-isomorphism,
2. the fiber of $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{Y}_{scaled}$ over $0 \neq \lambda \in \mathbb{A}^1$ coincides with f,
3. the fiber of $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{Y}_{scaled}$ over $0 = \lambda \in \mathbb{A}^1$ is the zero section.

We will follow [GRII]. The idea is to construct $\mathbb{A}^1$-family of groupoids $\mathcal{R}^{\bullet}_{scaled} \in \mathrm{PreSt}^{laft-def}$ deforming the groupoid $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \rightrightarrows \mathcal{X}$ to $\mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y}))) \rightrightarrows \mathcal{X}$.

The construction is given by

$$\mathcal{R}^{\bullet}_{scaled} = (\mathrm{Weil}^{\mathrm{Bifurc}^{\bullet}}_{\mathbb{A}^1}(\mathcal{X} \times \mathrm{Bifurc}^{\bullet}) \times_{\mathrm{Weil}^{\mathrm{Bifurc}^{\bullet}}_{\mathbb{A}^1}(\mathcal{Y} \times \mathrm{Bifurc}^{\bullet})} (\mathcal{Y} \times \mathbb{A}^1))^{\wedge}_{\mathcal{X} \times \mathbb{A}^1}, \tag{1.2}$$

for some $\mathbb{A}^1$-family of groupoids $\mathrm{Bifurc}^{\bullet} \in (\mathrm{AffSch}^{cl})^{\mathrm{op}}$.

Concretely, for $\lambda \in \mathbb{A}^1$ we have

$$(\mathcal{R}^{\bullet}_{scaled})_{\lambda} = (\mathrm{Maps}((\mathrm{Bifurc}^{\bullet})_{\lambda}, \mathcal{X}) \times_{\mathrm{Maps}((\mathrm{Bifurc}^{\bullet})_{\lambda}, \mathcal{Y})} \mathcal{Y})^{\wedge}_{\mathcal{X}}. \tag{1.3}$$

From this formula we see that $\mathrm{Bifurc}^{\bullet} \in (Sch^{cl,\mathrm{aff}})^{\mathrm{op}}$ should be
$\mathrm{Spec}(k[u]) \underset{\epsilon \mapsto -u}{\overset{\epsilon \mapsto u}{\rightrightarrows}} \mathrm{Spec}(k[u,\epsilon]/(u-\epsilon)(u+\epsilon))$.

1.2. **Digression: (lax)-equivariance.** It turns out that $\mathcal{Y}_{scaled}$ carries a lax-equivariant structure with respect to the action of the monoid $\mathbb{A}^1$ (via multiplication). In this subsection we discuss generalities on (lax)-equivariance.

Let G be a monoid, C_1, C_2 be categories with an action of G. Suppose we have $\phi : C_1 \to C_2$. In this context we have a familiar notion of lax-equivariance:

Definition 1.3. Right-lax (left-lax) equivariant structure on Φ w.r.t. G is a homotopy coherent system of assignments

$$g \circ \Phi \to \Phi \circ g \ (\Phi \circ g \to g \circ \Phi)$$

compatibe with monoid structure.

We say that Φ is strictly equivariant if these maps are equivalences.

Let now $\mathcal{G}$ be a monoid prestack. let $\mathcal{C}_1$ and $\mathcal{C}_2$ be functors

$$\mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Cat}$$

with a pointwise action of $\mathcal{G}$. Suppose we have a natural transformation $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$.

Definition 1.4. Datum of right-lax (left-lax, strict) equivariance on Φ w.r.t. $\mathcal{G}$ is a compatible system of right-lax (left-lax, strict) equivariance structures on $\Phi(S) : \mathcal{C}_1(S) \to \mathcal{C}_2(S)$ for $S \in \mathrm{AffSch}$.

Remark 1.5. When $\mathcal{C}_1 = \mathcal{X}$ is a prestack we can view Φ as $\mathcal{C}_2(\mathcal{X})$, where we view $\mathcal{C}_2$ as a functor $\mathcal{C}_2 : \mathrm{PreSt}^{\mathrm{op}} \to \mathrm{Cat}$ via right Kan extension along $\mathrm{AffSch}^{\mathrm{op}} \hookrightarrow \mathrm{PreSt}^{\mathrm{op}}$.

We denote the category of right-lax (left-lax, strict-) equivariant Φ by $\mathcal{C}_2(\mathcal{X})^{\mathcal{G}_{\text{right-lax}}}$ ($\mathcal{C}_2(\mathcal{X})^{\mathcal{G}_{\text{left-lax}}}$, $\mathcal{C}_2(\mathcal{X})^{\mathcal{G}_{\text{lax}}}$).

Example. *$\mathcal{C}_2 = C \otimes \mathrm{QCoh}(-)$ with the trivial action of $\mathcal{G}$. In the case when $C = \mathrm{Vect}$ this is the familiar notion of equivariant quasi-coherent sheaf.*

Example. *$\mathcal{C}_2 = C \otimes \mathrm{IndCoh}(-)$ with the trivial action of $\mathcal{G}$.*

Example. *$\mathcal{C}_2 = \mathrm{PreSt}_{/-}$ with the trivial action of $\mathcal{G}$. Then for $\mathcal{Y} \in \mathrm{PreSt}$ the element of $\mathcal{C}_2(\mathcal{Y})^{\mathcal{G}_{\text{right-lax}}}$ is the data of a prestack $\mathcal{X} \to \mathcal{Y}$ plus a map $\mathcal{X}_{pr} \to \mathcal{X}_{act}$ and higher compatibilities, where the source and the target are defined as*

$$\begin{array}{ccc} \mathcal{X}_{pr} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{G} \times \mathcal{Y} & \xrightarrow[pr]{} & \mathcal{Y} \end{array}$$

and

$$\begin{array}{ccc} \mathcal{X}_{act} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{G} \times \mathcal{Y} & \xrightarrow[act]{} & \mathcal{Y}. \end{array}$$

Lemma 1.6. *The groupoid $\mathrm{Bifurc}^\bullet$ upgrades to an object $\mathrm{Bifurc}^\bullet \in (\mathrm{AffSch}_{/\mathbb{A}^1}^{\mathbb{A}^1_{\text{right-lax}}})^{\mathrm{op}}$.*

Corollary 1.7. *The prestack $\mathcal{Y}_{scaled} \in \mathrm{PreSt}^{laft-def}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1}$ upgrades to an object of $((\mathrm{PreSt}^{laft-def}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1})^{nil-iso})^{\mathbb{A}^1_{\text{left-lax}}}$.*

1.3. **The special case.** From now on let X be a smooth proper curve. In this subsection we specialize the above discussion to the following situation .Let $\mathcal{X} = S(Z) \times Ran$ for a sectionally laft prestack $Z \to X_{dR}$. Here $S(Z)(T) := \mathrm{Maps}_{X_{dR}}(T \times X_{dR}, Z)$ for any $T \in \mathrm{AffSch}$.

As a result of the previous subsections we get a functor

$$\mathrm{DefNorm} : (\mathrm{PreSt}_{laft-def}{}^{nil-iso}_{S(Z)\times Ran/})_{/Ran} \to ((\mathrm{PreSt}_{laft-def}{}^{nil-iso}_{S(Z)\times Ran\times\mathbb{A}^1/})_{/Ran\times\mathbb{A}^1})^{\mathbb{A}^1_{\text{left-lax}}}$$

sending $S(Z) \times Ran \to \mathcal{Y}$ to $\mathcal{Y}_{scaled}$.

Post-composing DefNorm with the functor

$$((\mathrm{PreSt}_{laft-def}{}^{nil-iso}_{S(Z)\times Ran\times\mathbb{A}^1/})_{/Ran\times\mathbb{A}^1})^{\mathbb{A}^1_{\text{left-lax}}} \to \mathrm{End}_{\mathrm{IndCoh}(Ran\times\mathbb{A}^1)}(\mathrm{IndCoh}(S(Z) \times Ran \times \mathbb{A}^1))^{\mathbb{A}^1_{\text{left-lax}}}$$

sending $f : S(Z) \times Ran \times \mathbb{A}^1 \to \mathcal{W}$ to $f^! f_*^{\mathrm{IndCoh}}$, we get

$$N : (\mathrm{PreSt}_{laft-def}{}^{nil-iso}_{S(Z)\times Ran/})_{/Ran} \to \mathrm{End}_{\mathrm{IndCoh}(Ran\times\mathbb{A}^1)}(\mathrm{IndCoh}(S(Z) \times Ran \times \mathbb{A}^1))^{\mathbb{A}^1_{\text{left-lax}}}.$$

The category $\mathrm{IndCoh}(\mathbb{A}^1)$ is dualizable, and therefore

$$\mathrm{IndCoh}(S(Z) \times Ran \times \mathbb{A}^1) \cong \mathrm{IndCoh}(S(Z) \times Ran) \otimes \mathrm{IndCoh}(\mathbb{A}^1) \cong \mathrm{IndCoh}(S(Z) \times Ran) \otimes \mathrm{QCoh}(\mathbb{A}^1),$$

and

$$\mathrm{IndCoh}(Ran \times \mathbb{A}^1) \cong \mathrm{IndCoh}(Ran) \otimes \mathrm{IndCoh}(\mathbb{A}^1) \cong \mathrm{IndCoh}(Ran) \otimes \mathrm{QCoh}(\mathbb{A}^1),$$

Lemma 1.8. *For A, B symmetric monoidal DG categories, $M \in B$-mod, we have*

$$\mathrm{End}_{B\ otimes A}(M \otimes A) \cong \mathrm{End}_B(M) \otimes A.$$

Then we can rewrite

$$N : (\mathrm{PreSt}_{laft-def}{}^{nil-iso}_{S(Z)\times Ran/})_{/Ran} \to (\mathrm{End}_{\mathrm{IndCoh}(Ran)}(\mathrm{IndCoh}(S(Z) \times Ran)) \otimes \mathrm{QCoh}(\mathbb{A}^1))^{\mathbb{A}^1_{\text{left-lax}}}.$$

1.4. **Digression: $\mathbb{A}^1$-equivariance and filtrations.** Let C be a DG category. Let $C^{\mathrm{Fil}} := \mathrm{Maps}(\mathbb{Z}, C)$ be the category of filtered objects. Here $\mathbb{Z}$ is viewed as an ordered set and hence a category.

Proposition 1.9. *There exists an equivalence*

$$C^{\mathrm{Fil}} \cong (C \otimes \mathrm{QCoh}(\mathbb{A}^1))^{\mathbb{G}_m},$$

where the action of $\mathbb{G}_m$ on $\mathbb{A}^1$ is via multiplication.

Proof. (Sketch) Reduce to the case $C = \mathrm{Vect}$. Then the functor $\mathrm{QCoh}(\mathbb{A}^1)^{\mathbb{G}_m} \to C^{\mathrm{Fil}}$ is given by $\mathcal{F} \mapsto (n \mapsto \Gamma(\mathbb{A}^1, \mathcal{F}(n \cdot \{0\}))^{\mathbb{G}_m})$. ☐

Under this identification we have

$$\begin{array}{ccccc}
 & \overset{\mathrm{colim}_{\mathbb{Z}}}{\nearrow} & C & \xrightarrow{\cong} & C \\
 & & \uparrow{\scriptstyle \mathrm{Oblv}^{\mathrm{Fil}}} & & \uparrow{\scriptstyle (\mathrm{Id}\otimes i_1^*)\circ \mathrm{Oblv}} \\
\mathrm{Maps}(\mathbb{Z}, C) & \xrightarrow{=:} & C^{\mathrm{Fil}} & \xrightarrow{\cong} & (C \otimes \mathrm{QCoh}(\mathbb{A}^1))^{\mathbb{G}_m} \\
\downarrow{\scriptstyle \mathrm{gr}} & & \downarrow{\scriptstyle \mathrm{gr}} & & \downarrow{\scriptstyle \mathrm{Id}\otimes i_0^*} \\
C^{\mathbb{Z}} & \xrightarrow[=:]{} & C^{gr} & \xrightarrow[\cong]{} & C^{\mathbb{G}_m}.
\end{array}$$

Recall that we consider $\mathbb{A}^1$ as a monoid under multiplication.

Lemma 1.10. *The forgetful functor*

$$(C \otimes \mathrm{QCoh}(\mathbb{A}^1))^{\mathbb{A}^1_{\text{left-lax}}} \to (C \otimes \mathrm{QCoh}(\mathbb{A}^1))^{\mathbb{G}_m}$$

is fully faithful and its essential image identifies with $C^{\mathrm{Fil},\geq 0} \subset C^{\mathrm{Fil}}$.

Lemma 1.11. *The forgetful functor*

$$C^{\mathbb{A}^1_{\text{left-lax}}} \to C^{\mathbb{G}_m}$$

is fully faithful and its essential image identifies with $C^{\mathrm{gr},\geq 0} \subset C^{\mathrm{gr}}$.

Remark 1.12. On $C^{\mathrm{Fil},\geq 0} \subset C^{\mathrm{Fil}}$ the functor gr is conservative.

1.5. **The special case: redux.** Using the results of the previous subsection we rewrite the functor N as

$$N : (\mathrm{PreSt}_{laft-def}{}^{nil-iso}_{S(Z)\times Ran/})_{/Ran} \to (\mathrm{End}_{\mathrm{IndCoh}(Ran)}(\mathrm{IndCoh}(S(Z) \times Ran)))^{\mathrm{Fil},\geq 0}.$$

Using Lemma 1.8 again we see that the target

$$(\mathrm{End}_{\mathrm{IndCoh}(Ran)}(\mathrm{IndCoh}(S(Z) \times Ran)))^{\mathrm{Fil},\geq 0} \cong (\mathrm{End}(\mathrm{IndCoh}(S(Z))) \otimes D(Ran))^{\mathrm{Fil},\geq 0}.$$

Summarizing, we get

(1.4)

$$
\begin{array}{ccccc}
 & & \mathrm{End}(\mathrm{IndCoh}(S(Z)))\otimes D(Ran) & \xrightarrow{\Gamma_{c,Ran}} & \mathrm{End}(\mathrm{IndCoh}(S(Z))) \\
 & \nearrow^{M:=(-)^!(-)_*^{\mathrm{IndCoh}}} & \uparrow \mathrm{Oblv}^{\mathrm{Fil}} & & \uparrow \mathrm{Oblv}^{\mathrm{Fil}} \\
(\mathrm{PreSt}_{laft-def}{}^{nil-iso}_{S(Z)\times Ran/})_{/Ran} & \xrightarrow{N} & (\mathrm{End}(\mathrm{IndCoh}(S(Z)))\otimes D(Ran))^{\mathrm{Fil},\geq 0} & \xrightarrow{\Gamma_{c,Ran}} & (\mathrm{End}(\mathrm{IndCoh}(S(Z))))^{\mathrm{Fil},\geq 0} \\
 & & \downarrow \mathrm{gr} & & \downarrow \mathrm{gr} \\
 & & \mathrm{End}(\mathrm{IndCoh}(S(Z)))\otimes D(Ran) & \xrightarrow{\Gamma_{c,Ran}} & \mathrm{End}(\mathrm{IndCoh}(S(Z))) \\
 & \searrow_{\mathrm{Sym}(T(S(Z)\times Ran/-))} & \uparrow \mathrm{Tens} & & \uparrow \mathrm{Tens} \\
 & & \mathrm{IndCoh}(S(Z))\otimes D(Ran) & \xrightarrow{\Gamma_{c,Ran}} & \mathrm{IndCoh}(S(Z))
\end{array}
$$

2. Statement of the main result.

Let $Z \in \mathrm{PreSt}_{/X_{dR}}$ be a sectionally laft prestack. Along with horizontal sections $S(Z)$ we have the puntured sections prestack living over Ran^{untl}:

Definition 2.1. Informally, the fiber of puntured sections prestack $\overset{\circ}{S}(Z)_{Ran^{\mathrm{untl}}}$ over $x_1, ..., x_n \in Ran^{\mathrm{untl}}(\mathbb{C})$ is given by

$$\overset{\circ}{S}(Z)_{\{x_1,...,x_n\}} = S(Z|_{\{x_1,...,x_n\}}).$$

Remark 2.2. We have a natural map $S(Z) \times Ran^{\mathrm{untl}} \to \overset{\circ}{S}(Z)_{Ran^{\mathrm{untl}}}$ given fiberwise by restriction.

The prestack $\overset{\circ}{S}(Z)_{Ran^{\mathrm{untl}}}$ is not laft so we introduce a laft prestack

Definition 2.3. $\overset{\circ}{S}(Z)^{\wedge}_{Ran^{\mathrm{untl}}} := (\overset{\circ}{S}(Z)_{Ran^{\mathrm{untl}}})^{\wedge}_{S(Z)\times Ran_X^{\mathrm{untl}}}$.

By construction the morphism

$$r^{\mathrm{untl}} : S(Z) \times Ran_X^{\mathrm{untl}} \to \overset{\circ}{S}(Z)^{\wedge}_{Ran^{\mathrm{untl}}}$$

is nil-isomorphism, hence pseudo-proper. Therefore $r^{\mathrm{untl}!}$ admits a left adjoint $r^{\mathrm{untl}}{}_*$ satisfying projection formula.

Definition 2.4. We define the infinitesimal Hecke monad

$$\mathcal{H}^{\mathrm{inf}}(Z)_{\mathrm{untl}} := (r^{\mathrm{untl}})^!(r^{\mathrm{untl}})_* \in \mathrm{Alg}(\mathrm{End}_{\mathrm{IndCoh}(Ran^{\mathrm{untl}})}(\mathrm{IndCoh}(S(Z) \times Ran^{\mathrm{untl}}))).$$

Remark 2.5. The exact same constructions make sense for Ran^{untl} replaces by Ran. Denote the corresponding infinitesimal monad by $\mathcal{H}^{\mathrm{inf}}(Z)$.

For any prestack $\mathcal{Y}$ consider $p_{\mathcal{Y},dR} : \mathcal{Y} \to \mathcal{Y}_{dR}$. This is nil-iso, so $(p_{\mathcal{Y},dR})^!$ admits a left adjoint $(p_{\mathcal{Y},dR})_*$ satisfying projectioon formula.

Definition 2.6. Denote the resulting monad $(p_{\mathcal{Y},dR})^! \circ (p_{\mathcal{Y},dR})_*$ by $\mathrm{Diff}_{\mathcal{Y}}$.

Remark 2.7. We have $\mathrm{Diff}_{\mathcal{Y}}\text{-mod}(\mathrm{IndCoh}(\mathcal{Y})) \cong D(\mathcal{Y})$.

We also have

$$
\begin{array}{ccc}
S(Z) \times Ran_X^{\mathrm{untl}} & \xrightarrow[p_{S(Z),dR}\times \mathrm{Id}]{} & S(Z)_{dR} \times Ran_X^{\mathrm{untl}} \\
 & {}_{r^{\mathrm{untl}}}\searrow \quad \nearrow & \\
 & \overset{\circ}{S}(Z)^{\wedge}_{Ran^{\mathrm{untl}}}. &
\end{array}
$$

and therefore we have a map of monads

$$\mathcal{H}^{\mathrm{inf}}(Z)_{\mathrm{untl}} \to \mathrm{Diff}_{S(Z)} \boxtimes \omega_{Ran_X^{\mathrm{untl}}}. \tag{2.1}$$

Applying (symmetric monoidal!) $\Gamma_{c,Ran^{\mathrm{untl}}}$ we get a map

$$\Gamma_{c,Ran^{\mathrm{untl}}}(\mathcal{H}^{\inf}(Z)_{\mathrm{untl}}) \to \mathrm{Diff}_{S(Z)} \in \mathrm{Alg}(\mathrm{End}(\mathrm{IndCoh}(S(Z)))). \tag{2.2}$$

Theorem 2.1. *If $Z \in \mathrm{PreSt}_{/X_{dR}}$ is sectionally laft, then (2.2) is an isomoprhism.*

This Theorem is the main goal of the talk. We will follow the proof in [R].

Remark 2.8. It suffices to prove the isomorphism between the underlying objects in $\mathrm{End}(\mathrm{IndCoh}(S(Z))$.

3. Proof of the main result.

3.1. **Step 1: Reduction to the statement about tangent complexes.** We want to employ results from the first section, therefore we reduce the statement to the non-unital one.

Recall that $\iota : Ran \to Ran^{\mathrm{untl}}$ is universally homologically cofinal, hence

$$\Gamma_{c,Ran^{\mathrm{untl}}} \cong \Gamma_{c,Ran} \circ \iota^!.$$

Note that $(\mathrm{Id} \otimes \iota^!)(\mathcal{H}^{\inf}(X)_{\mathrm{untl}}) \cong \mathcal{H}^{\inf}(Z)$. Hence we need to show that

$$\Gamma_{c,Ran}(\mathcal{H}^{\inf}(Z)) \to \mathrm{Diff}_{S(Z)}$$

arizing by adjunction from

$$\mathcal{H}^{\inf}(Z) \to \mathrm{Diff}_{S(Z)} \boxtimes \omega_{Ran} \tag{3.1}$$

is an isomorphism.

Recall the functor

$$M := (-)^!(-)_*^{\mathrm{IndCoh}} : (\mathrm{PreSt}_{laft-def}{}^{nil-iso}_{S(Z)\times Ran/})_{/Ran} \to \mathrm{End}(\mathrm{IndCoh}(S(Z))) \otimes D(Ran).$$

By definition the map (3.1) coincides with

$$M(\overset{\circ}{S}(Z)^{\wedge}_{Ran} \to S(Z)_{dR} \times Ran).$$

Chasing (1.4) we see that it suffices to show that

$$N((\overset{\circ}{S}(Z)^{\wedge}_{Ran} \to S(Z)_{dR} \times Ran)$$

is an isomorphism. Or, equivalently, that

$$\Gamma_{c,Ran} \circ N((\overset{\circ}{S}(Z)^{\wedge}_{Ran} \to S(Z)_{dR} \times Ran)$$

is an isomorphism. Since the filtrations are non-negative, the functors gr in (1.4) are conservative. Therefore it suffices to show that

$$\Gamma_{c,Ran}(\mathrm{Sym}(T(S(Z) \times Ran/-)))(\overset{\circ}{S}(Z)^{\wedge}_{Ran} \to S(Z)_{dR} \times Ran) \tag{3.2}$$

is an isomorphism in $\mathrm{IndCoh}(S(Z))$.

However, by universal homological cofinality this is equivalent to

$$\Gamma_{c,Ran^{\mathrm{untl}}}(\mathrm{Sym}(T(S(Z) \times Ran^{\mathrm{untl}}/-)))(\overset{\circ}{S}(Z)^{\wedge}_{Ran^{\mathrm{untl}}} \to S(Z)_{dR} \times Ran^{\mathrm{untl}}) \tag{3.3}$$

being an isomorphism. But since $\Gamma_{c,Ran^{\mathrm{untl}}}$ is symmetric monoidal it suffices to show that

$$\Gamma_{c,Ran^{\mathrm{untl}}}(T(S(Z)\times Ran^{\mathrm{untl}}/\ \overset{\circ}{S}(Z)^{\wedge}_{Ran^{\mathrm{untl}}})) \to \Gamma_{c,Ran^{\mathrm{untl}}}(T(S(Z)\times Ran^{\mathrm{untl}}/S(Z)_{dR}\times Ran^{\mathrm{untl}})). \tag{3.4}$$

So since $T(S(Z)_{dR} \times Ran^{\mathrm{untl}}) \cong 0$ we reduced the main theorem to the following statement:

Theorem 3.2. *Let $Z \in \mathrm{PreSt}_{/X_{dR}}$ be sectionally laft. Then*

$$\Gamma_{c,Ran^{\mathrm{untl}}} \circ (r^{\mathrm{untl}})^!(T(\overset{\circ}{S}(Z)^{\wedge}_{Ran^{\mathrm{untl}}})) = 0.$$

3.3. **Step 2: reduction of theorem 3.2 to statements about $\Theta(Z)$.** Recall the definition of the antecedent of the tangent complex:

Definition 3.1. For $x : S \to S(Z)$, $S \in \mathrm{AffSch}$ the sheaves
$$\mathbb{D}^{SV} \circ \mathrm{Oblv}^{\mathrm{fake}} \circ \mathrm{ev}_x^{\sharp} T^* Z \in \mathrm{IndCoh}(S \times X_{dR}),$$
where $\mathrm{ev}_x : S \times X_{dR} \to Z$ is the section corresponding to x, assemble to $\Theta(Z) \in \mathrm{IndCoh}(S(Z) \times X_{dR})$.

Here is the main property of this sheaf that we are going to use:

Proposition 3.2. *$T(S(Z)) \cong (\mathrm{id} \times p_{X_{dR}})_*^{\mathrm{IndCoh}} \Theta(Z) \cong (\mathrm{id} \times p_{X\,dR,*})\Theta(Z)$, where $p_{X_{dR}} : X_{dR} \to \mathrm{pt}$ and $p_X : X \to \mathrm{pt}$.*

The proof of Theorem 3.2 will consist of the following two assertions:

Proposition 3.3. *Let $Z \in \mathrm{PreSt}_{/X_{dR}}$ be sectionally laft. Then*
$$\Gamma_{c,Ran^{\mathrm{untl}}}(T(S(Z) \times Ran^{\mathrm{untl}} / \overset{\circ}{S}(Z)^{\wedge}_{Ran^{\mathrm{untl}}})) \in \mathrm{IndCoh}(S(Z)) \otimes D(Ran^{\mathrm{untl}})$$
lies in the essential image of $\mathrm{id} \otimes (\iota_X)_!$, where $(\iota_X)_! : D(X) \to D(Ran^{\mathrm{untl}})$.

Proposition 3.4. *Under the isomorphism in Proposition 3.2 the map*
$$\mathrm{id} \otimes (\iota_X)^!(T(S(Z) \times Ran^{\mathrm{untl}} / \overset{\circ}{S}(Z)^{\wedge}_{Ran^{\mathrm{untl}}})) \to \mathrm{id} \otimes (\iota_X)^!(T(S(Z) \times Ran^{\mathrm{untl}})) \cong T(S(Z)) \boxtimes \omega_X$$
identifies with
$$\Theta(Z) \to (\mathrm{id} \otimes (p_X)^!) \circ (\mathrm{Id} \otimes p_{X\,dR,*})\Theta(Z)$$
given by adjunction.

We now deduce the proof of Theorem 3.2 from these two propositions. Consider the diagram

$$\begin{array}{ccc}
(\mathrm{Id} \otimes \Gamma_{c,Ran^{\mathrm{untl}}})(T(S(Z) \times Ran^{\mathrm{untl}} / \overset{\circ}{S}(Z)^{\wedge}_{Ran^{\mathrm{untl}}})) & \longrightarrow & (\mathrm{Id} \otimes \Gamma_{c,Ran^{\mathrm{untl}}})(T(S(Z) \times Ran^{\mathrm{untl}}) \\
\cong \uparrow \mathrm{Pr}.3.3 & & \uparrow \\
(\mathrm{Id} \otimes \Gamma_{c,Ran^{\mathrm{untl}}}) \circ (\mathrm{id} \otimes (\iota_X)^!)(T(S(Z) \times Ran^{\mathrm{untl}} / \overset{\circ}{S}(Z)^{\wedge}_{Ran^{\mathrm{untl}}}) & \xrightarrow[pr]{} & (\mathrm{Id} \otimes \Gamma_{c,Ran^{\mathrm{untl}}}) \circ (\mathrm{id} \otimes (\iota_X)^!)(T(S(Z) \times Ran^{\mathrm{untl}})
\end{array}$$

By Proposition 3.4the lower composition identifies with

(3.5)
$$(\mathrm{Id} \otimes \Gamma_{c,X})\Theta(Z) \to (\mathrm{Id} \otimes \Gamma_{c,X}) \circ (\mathrm{Id} \otimes (p_X)^!) \circ (\mathrm{Id} \otimes (p_X)_{dR,*})\Theta(Z) \to (\mathrm{Id} \otimes \Gamma_{c,Ran^{\mathrm{untl}}}) \circ (\mathrm{Id} \otimes (p_{Ran^{\mathrm{untl}}})^!) \circ (\mathrm{Id} \otimes (p_X)_{dR,*})\Theta(Z).$$

But counit of the adjoint pair $(\Gamma_{c,Ran^{\mathrm{untl}}}), (p_{Ran^{\mathrm{untl}}})^!)$ is identity, hence it suffices to show that the comoposition of the last map of (3.5) with
$$(\mathrm{Id} \otimes \Gamma_{c,Ran^{\mathrm{untl}}}) \circ (\mathrm{Id} \otimes (p_{Ran^{\mathrm{untl}}})^!) \circ (\mathrm{Id} \otimes (p_X)_{dR,*})\Theta(Z) \to (\mathrm{Id} \otimes (p_X)_{dR,*})\Theta(Z)$$
is an isomorphism. This composition identifies with
$$(\mathrm{Id} \otimes (p_X)_{dR,*})\Theta(Z) \to (\mathrm{Id} \otimes (p_X)_{dR,*}) \circ (\mathrm{Id} \otimes (p_{Ran^{\mathrm{untl}}})^!) \circ (\mathrm{Id} \otimes (p_X)_{dR,*})\Theta(Z) \to (\mathrm{Id} \otimes (p_X)_{dR,*})\Theta(Z),$$
which is isomorphism by adjunction axioms. Hence we proved Theorem 3.2 and therefore Theorem 2.1.

References

[GRII] "A study in derived algebraic geometry Volume II: Deformations, Lie theory and formal geometry" D. Gaitsgory, N. Rozenblyum

[R] "Connections on moduli spaces and infinitesimal Hecke modifications" N. Rozenblyum