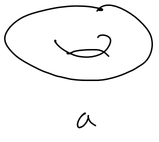
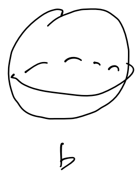
Ran Space

Let X be a homotopy type (i.e. a space).

$\mathrm{Ran}^{\mathrm{untl}}(X)$:= poset of finite subsets of $\pi_0(X)$

$$\mathrm{Ran}(X) := \mathrm{Ran}^{\mathrm{untl}}(X)^{\mathrm{grpd}} \setminus \{\emptyset\}$$

Example : $X =$ 

$\sqcup$ 

$$\mathrm{Ran}^{\mathrm{untl}}(X) = \begin{tikzcd} & \{a,b\} & \\ \{a\} \arrow[ur] & & \{b\} \arrow[ul] \\ & \{\emptyset\} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$$\mathrm{Ran}(X) = \{\{a\}, \{b\}, \{a,b\}\}$$

We have $\mathrm{Ran}^{\mathrm{untl}}(X) = \mathrm{Fin}_{/X}[(\mathrm{Fin}^{s}_{/X})^{-1}]$,

$$\mathrm{Ran}(X) = \operatorname*{colim}_{I \in (\mathrm{Fin}^{s})^{op}} X^{I}.$$

Ran PreStack

Now let X be a separated scheme.

The prestack $\mathrm{Ran}(X)(S) := \mathrm{Ran}(X_{dR}(S)) = \mathrm{Ran}(X({}^{red}S))$.

The lax prestack $\mathrm{Ran}^{untl}(X)(S) := \mathrm{Ran}^{untl}(X_{dR}(S)) = \mathrm{Ran}(X({}^{red}S))$.

When X is locally of finite type, $\mathrm{Ran}(X)$ and $\mathrm{Ran}^{untl}(X)$ are laft

$$\Longrightarrow \mathrm{IndCoh}(\mathrm{Ran}(X)),\ D(\mathrm{Ran}(X))\ ;\ \mathrm{IndCoh}(\mathrm{Ran}^{untl}(X)),\ D(\mathrm{Ran}^{untl}(X))$$

are well defined.

Since $\mathrm{Ran}(X) \xrightarrow{\sim} \mathrm{Ran}(X)_{dR}$, $\mathrm{Ran}^{untl}(X) \xrightarrow{\sim} \mathrm{Ran}^{untl}(X)_{dR}$

we have

$$D(\mathrm{Ran}(X)) \xrightarrow{\sim} \mathrm{IndCoh}(\mathrm{Ran}(X))\ ;\ D(\mathrm{Ran}^{untl}(X)) \xrightarrow{\sim} \mathrm{IndCoh}(\mathrm{Ran}^{untl}(X)).$$

$$M \in D(\mathrm{Ran}^{untl}(X)) \rightsquigarrow \forall I \in \mathrm{Fin},\ M_I \in D(X^I)$$

$$\forall I \xrightarrow{f} J,\ \Delta_f^!(M_I) \longrightarrow M_J \quad \in D(X^J)$$

s.t. this is an isom whenever f surjective

+ higher homotopy coherences

$M \in D(\mathrm{Ran}(X)) \rightsquigarrow \forall I \in \mathrm{Fin}^s, \quad M_I \in D(X^I)$

$\forall I \xrightarrow{f} J, \quad \Delta_f^!(M_I) \xrightarrow{\sim} M_J \in D(X^J)$

+ higher homotopy coherences

<u>Pseudo-Properness</u> Def'n $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$; $(\mathrm{PreStk}_{\mathrm{aft}})^{\mathrm{ps-pr}}_{/S} \subseteq (\mathrm{PreStk}_{\mathrm{aft}})_{/S}$

:= smallest full subcat closed under colimits containing $X \longrightarrow S$

s.t. X is a scheme, $X \to S$ is proper.

Def'n

For $\mathcal{Y}$ laft prestack, $\mathrm{PreStk}^{\mathrm{ps-pr}}_{/\mathcal{Y}} \subseteq \mathrm{PreStk}_{/\mathcal{Y}}$

:= full subcat of $\mathcal{X} \to \mathcal{Y}$ s.t. $\mathcal{X} \underset{\mathcal{Y}}{\times} S \longrightarrow S$ is ps-pr $\forall \begin{array}{c} S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}} \\ \downarrow \\ \mathcal{Y} \end{array}$.

<u>Rmk</u> $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ ps-pr $\Rightarrow$ $f^!$ has "well behaved" left adj $f_!$

($f_!$ compatible w/ base change, satisfies projection formula.)

Def'n

For $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ map of laft categorical prestacks, we say it is ps-pr if

- f is level-wise a cocartesian fibration in gpds
- $\forall \begin{array}{c} S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}} \\ \downarrow \\ \mathcal{Y} \end{array}$, $\mathcal{X} \underset{\mathcal{Y}}{\times} S \longrightarrow S$ is ps-pr.

<u>Stability of Ps-Pr</u>

- stable under colimits (by def'n), base change, compositions, products

Prop

a) $\forall I$, $X^I_{dR} \longrightarrow \mathrm{Ran}(X)$ is ps-pr

b) $\mathrm{Ran}(X) \xrightarrow{\Delta} \mathrm{Ran}(X) \times \mathrm{Ran}(X)$ is ps-pr

c) $\mathrm{Ran}(X) \times \mathrm{Ran}(X) \xrightarrow{\cup} \mathrm{Ran}(X)$ is ps-pr

Pf

a) If $I \longrightarrow \mathrm{Sch}$ w/ $F(i) \longrightarrow F(j)$ proper, $F(i) \longrightarrow \mathrm{colim}\, F$ is ps-pr.

b)
$$\begin{tikzcd}
X^I_{dR} \arrow[r] \arrow[dr, bend right, "\text{proper}"'] & \mathrm{Ran}(X) \arrow[r, "\Delta"] & \mathrm{Ran}(X) \times \mathrm{Ran}(X) \\
 & X^I_{dR} \times X^I_{dR} \arrow[ur, "\text{ps-pr} \times \text{ps-pr}"'] &
\end{tikzcd}$$

c)
$$\begin{tikzcd}
X^I_{dR} \times X^J_{dR} \arrow[r] \arrow[d, "\sim"] & \mathrm{Ran}(X) \times \mathrm{Ran}(X) \arrow[r, "\cup"] & \mathrm{Ran}(X) \\
X^{I \sqcup J}_{dR} \arrow[urr, "\text{ps-pr}"'] & &
\end{tikzcd}$$

$\square$

Compact Generation

Def'n Let $\mathcal{C}$ be a $\overset{\text{(cocomplete)}}{\text{DG Category}}$.

- We say $c \in \mathcal{C}$ is compact if $\mathrm{Maps}(c, -)$ is continuous. Denote by $\mathcal{C}^c$ the subcategory of compact objects.
- For $\mathcal{D} \subseteq \mathcal{C}$, we say $\mathcal{D}$ generates $\mathcal{C}$ if $\mathrm{Maps}(d, c) = 0 \ \forall d \in \mathcal{D}$ implies $c = 0$ $\Leftrightarrow$ Smallest cocomplete subcategory of $\mathcal{C}$ containing $\mathcal{D}$ is $\mathcal{C}$.
- $\mathcal{C}$ is said to be compactly generatd if $\mathcal{C}^c$ generates $\mathcal{C}$.

If $\mathcal{C}$ is compactly generated, $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}^c)$ so $\mathcal{C}$ is dualizable w/

$$\mathcal{C}^\vee = \mathrm{Ind}(\mathcal{C}^c)^\vee = \mathrm{Ind}(\mathcal{C}^{c,op}).$$

Fact Let $\Psi : I \to \mathrm{DGCat}_{cont}$. Suppose $\mathcal{C}_i \longrightarrow \mathcal{C}_j$ admits a cts right adjoint $\forall\, i \to j$. Denote $\Phi : I^{op} \longrightarrow \mathrm{DGCat}_{cont}$ the diagram of right adjoints. Then $\operatorname{colim}_I \mathcal{C}_i \simeq \lim_{I^{op}} \mathcal{C}_i$.
Furthermore, if $\mathcal{C}_i$ compactly generated $\forall i$, $\operatorname{colim}_I \mathcal{C}_i$ is compactly generated.

cptly generated b/c X^I is a scheme

$$\text{Since } \mathrm{Ran}(X) = \operatorname*{colim}_{I \in \mathrm{Fin}^{s,op}} X^I, \quad D(\mathrm{Ran}(X)) = \lim_{I \in \mathrm{Fin}^s} \left(\overset{\downarrow}{D(X^I)}, f^!\right)$$

$$= \operatorname*{colim}_{I \in \mathrm{Fin}^{s,op}} \left(D(X^I), \underset{\uparrow}{f_!}\right),$$

defined b/c for $I \twoheadrightarrow J$, $X^J \to X^I$ is proper

$D(\mathrm{Ran}(X))$ is cptly generated.

<u>Lemma</u> Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ continuous, $F \dashv G$.

- G cts $\Rightarrow$ F preserves compacts
  If $\mathcal{C}$ is cptly generated, this is $\Leftrightarrow$.
- G conservative $\Leftrightarrow$ (ess. image of) F generates
- Thus, if $\mathcal{C}$ is cptly generated and G is cts + conservative
  then $\mathcal{D}$ is cptly generated.

<u>Thm</u> $D(\mathrm{Ran}^{\mathrm{untl}}(X))$ is cptly generated ($\Rightarrow$ dualizable).

<u>Pf</u> Let $\mathrm{Ran}_{\emptyset}(X) := (\mathrm{Ran}^{\mathrm{untl}}(X))^{\mathrm{grpd}} = \mathrm{Ran}(X) \sqcup \{\emptyset\}$.

$D(\mathrm{Ran}_{\emptyset}(X))$ is compactly generated.

Now consider $i : \mathrm{Ran}_{\emptyset}(X) \longrightarrow \mathrm{Ran}^{\mathrm{untl}}(X)$

$i^! : D(\mathrm{Ran}^{\mathrm{untl}}(X)) \longrightarrow D(\mathrm{Ran}_{\emptyset}(X))$ is cts + conservative.

It suffices to show $i^!$ admits a left adjoint.

$$\begin{tikzcd}
\mathrm{Ran}^{\rightarrow}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Ran}_{\emptyset}(X) \times \mathrm{Ran}^{\mathrm{untl}}(X) \arrow[d, "\cup \times p_2"] \\
\mathrm{Ran}^{\mathrm{untl}}(X) \arrow[r, "\Delta"] & \mathrm{Ran}^{\mathrm{untl}}(X) \times \mathrm{Ran}^{\mathrm{untl}}(X)
\end{tikzcd}$$

$$\mathrm{Ran}^{\rightarrow}(X)(S) = \left\{ (I, J) \;\middle|\; \begin{matrix} I \in \mathrm{Ran}_{\emptyset}(X)(S) \\ J \in \mathrm{Ran}^{\mathrm{untl}}(X)(S) \end{matrix} \quad I \subseteq J \right\}$$

$$\begin{tikzcd}
 & & \mathrm{Ran}^{\rightarrow}(X) \arrow[ddll, "\xi"'] \arrow[ddrr, "\phi"] & & \\
 & & I \subseteq J \arrow[dl, mapsto] \arrow[dr, mapsto] & & \\
\mathrm{Ran}_{\emptyset}(X) & I & & J & \mathrm{Ran}^{\mathrm{untl}}(X)
\end{tikzcd}$$

ϕ is pseudo-proper, so $\phi_! \dashv \phi^!$ is defined.

We claim that $i_! := \phi_! \circ \xi^! : D(\mathrm{Ran}^{\mathrm{untl}}(X)) \longrightarrow D(\mathrm{Ran}_{\varnothing}(X))$ is the left adjoint of $i^!$:

- i factors as $\mathrm{Ran}_{\varnothing}(X) \xrightarrow{\nu} \mathrm{Ran}^{\rightarrow}(X) \xrightarrow{\phi} \mathrm{Ran}^{\mathrm{untl}}(X)$

$$I \longmapsto (I, I)$$
$$(I, J) \longmapsto J$$

so $i^! = \nu^! \circ \phi^!$.

Now it suffices to show $\xi^! \dashv \nu^!$. This follows from $\nu \dashv \xi$ as maps of lax prestk.

□

Universal Homological Contractibility

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor w/ $\mathcal{C}$ an ∞-cat, $\mathcal{D}$ an ∞-gpd.

Then fibers of F are contractible $\iff$

$\forall \mathcal{E}$ $\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ is fully faithful.

Def'n $\mathcal{Y}_1 \xrightarrow{f} \mathcal{Y}_2$ $\mathcal{Y}_1 \in \mathrm{laxPreStk}$, $\mathcal{Y}_2 \in \mathrm{PreStk}$.

We say f is UHC if $\forall S \in \mathrm{Sch}_{/\mathcal{Y}_2}$

$$D(S) \xrightarrow{f_S^!} D(S \times_{\mathcal{Y}_2} \mathcal{Y}_1)$$

is fully faithful ($\Longrightarrow D(\mathcal{Y}_2) \xrightarrow{f^!} D(\mathcal{Y}_1)$ is f.f.)

Rmk If $\forall S \in Sch$, $\mathcal{Y}_1(S) \longrightarrow \mathcal{Y}_2(S)$ has contractible fibers, $\mathcal{Y}_1 \xrightarrow{f} \mathcal{Y}_2$ is UHC.

Thm $Ran^{untl}(X)$ is UHC since $Ran^{untl}(X_{dR}(S))$ is contractible.

What if $\mathcal{D}$ isn't an ∞-grpd?

Defn $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is called contractible if for any $d \xrightarrow{\alpha} d'$,

$$Factor_F(\alpha) = \{ d \to F(c) \to d' \}$$

is contractible.

In general, F contractible $\Longleftrightarrow$ $\overset{\forall \mathcal{E}}{Fun(\mathcal{D}, \mathcal{E})} \longrightarrow Fun(\mathcal{C}, \mathcal{E})$ fully faithful

(If F is a co/cartesian fibration, $\Longleftrightarrow$ contractible fibers)

A weaker notion: instead of asking for $Fun(\mathcal{D}, \mathcal{E}) \longrightarrow Fun(\mathcal{C}, \mathcal{E})$ f.f.,

$$\Longleftrightarrow Maps_{Fun(\mathcal{D},\mathcal{E})}(\Phi_1, \Phi_2) \xrightarrow{\sim} Maps_{Fun(\mathcal{C},\mathcal{E})}(\Phi_1 \circ F, \Phi_2 \circ F),$$

Can ask only for this to hold when Φ_1 / Φ_2

maps $\mathcal{D}$ to $\mathcal{E}^{grpd}$ $\Longleftrightarrow$ $\mathcal{C}_{/d} / \mathcal{C}_{d/}$ contractible "right cofinal" / "left cofinal".

<u>When</u> $\mathcal{D}$ is a gpd, F contractible $\Leftrightarrow$ F left/right cofinal $\Leftrightarrow$ contractible fibers

<u>Def'n</u> Map of laxPreStk $\mathcal{Y}_1 \xrightarrow{f} \mathcal{Y}_2$.

$S \in Sch$, $\alpha : y_2' \longrightarrow y_2''$ in $\mathcal{Y}_2(S)$

$\mathrm{Factor}_f(\alpha) \in \mathrm{laxPreStk}_{/S}$

$$\mathrm{Factor}_f(\alpha)\left(\begin{matrix}\tilde{S}\\ \downarrow \\ S\end{matrix}\right) = \left\{ y_2'|_{\tilde{S}} \to f(y_1) \to y_2''|_{\tilde{S}} \;\middle|\; y_1 \in \mathcal{Y}_1(S) \right\}.$$

f is UHC if $\forall S, \alpha$, $\mathrm{Factor}_f(\alpha) \longrightarrow S$ is UHC.

<u>Def'n</u> Map of laxPreStk $\mathcal{Y}_1 \xrightarrow{f} \mathcal{Y}_2$.

$S \in Sch$, $y_2 \in \mathcal{Y}_2(S)$ $\rightsquigarrow$ $(\mathcal{Y}_1)_{y_2/} \in \mathrm{laxPreStk}_{/S}$

$$(\mathcal{Y}_1)_{y_2/}\left(\begin{matrix}\tilde{S}\\ \downarrow \\ S\end{matrix}\right) = \left\{ (y_1 \in \mathcal{Y}_1(\tilde{S}), \; y_2|_{\tilde{S}} \to f(y_1)) \right\}$$

f UHLC if $\forall (S, y_2)$, $(\mathcal{Y}_1)_{y_2/} \longrightarrow S$ is UHC.

Rmk If f is value-wise Contractible / left cofinal / right cofinal

f is UHC / UHLC / UHRC.

Rmk f UHC / UHLC / UHRC $\Rightarrow$

$$f^! \text{ fully faithful} \;\Big/\; \begin{array}{c} \forall\, \mathcal{F} \in D(\mathcal{Y}_2),\ \mathcal{G} \in D_{str}(\mathcal{Y}_1) \\ \mathrm{Maps}_{\mathcal{Y}_2}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Maps}_{\mathcal{Y}_1}(f^!\mathcal{F}, f^!\mathcal{G}) \end{array} \Big/ \leftrightarrow \quad .$$

Thm $$\mathrm{Ran}^{untl}(X) \xrightarrow{\Delta} \mathrm{Ran}^{untl}(X) \times \mathrm{Ran}^{untl}(X)$$

is valuewise left cofinal $\Rightarrow$ UHLC.

Linear Factorization Sheaves

Thm X a separated laft scheme.

$$i : X \longrightarrow \mathrm{Ran}^{untl}(X).$$

$\exists\ i_! \dashv i^!$.

Pf

$$\begin{tikzcd}
\mathrm{Ran}^{untl}_*(X) \arrow[r, "pr"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[dd, bend right=60, "q"'] & X_{dR} \arrow[d, "\text{ps-pr}"] \\
\mathrm{Ran}^{\to}(X) \arrow[r, "\xi"] \arrow[d, "\phi\quad \text{ps-pr}"] & \mathrm{Ran}_{\emptyset}(X) \\
\mathrm{Ran}^{untl}(X) &
\end{tikzcd}$$

$$\begin{tikzcd}
X_{dR} \arrow[r, "i_0"] \arrow[rr, bend right, "i"'] & \mathrm{Ran}^{untl}_* \arrow[r, "q"] & \mathrm{Ran}^{untl}(X)
\end{tikzcd}$$

$i_0 \dashv pr \rightsquigarrow pr^! \dashv i_0^!$; q ps-pr $\Rightarrow \exists\, q_! \dashv q^!$.

$\therefore\ \exists\ i_! = q_! \circ pr^! \dashv i_0^! \circ q^! = i^!$. $\square$

Thm $i_!$ is fully faithful.

Pf

$$\begin{tikzcd}
X_{dR} \arrow[r, "i_0"] \arrow[d, "id"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ran}^{untl}_{*}(X) \arrow[d, "q"] \\
X_{dR} \arrow[r, "i"] & \mathrm{Ran}^{untl}(X)
\end{tikzcd}$$

q ps-pr $\Rightarrow i^! q_! = i_0^! \Rightarrow i^! \circ i_! = i^! \circ q_! \circ pr^!$

$= i_0^! pr^! = id$

$\square$

Now we will describe the ess. im. of $i_!$.

Def'n $\cup : \mathrm{Ran}^{untl}(X) \times \mathrm{Ran}^{untl}(X) \longrightarrow \mathrm{Ran}^{untl}(X)$.

Since $I, J \subseteq I \cup J$, there are maps

$p_1 \longrightarrow \cup$, $p_2 \longrightarrow \cup$ in $\mathrm{Maps}(\mathrm{Ran}^{untl}(X) \times \mathrm{Ran}^{untl}(X), \mathrm{Ran}^{untl}(X))$

$\rightsquigarrow$ for $\mathcal{F} \in D(\mathrm{Ran}^{untl}(X))$, $p_1^! \mathcal{F} \rightarrow \cup^! \mathcal{F}$, $p_2^! \mathcal{F} \longrightarrow \cup^! \mathcal{F}$

$\rightsquigarrow p_1^! \mathcal{F} \oplus p_2^! \mathcal{F} \longrightarrow \cup^! \mathcal{F}$. ☆

$LFS(X) \subseteq D(\mathrm{Ran}^{untl}(X))$

$:= \mathcal{F}$ s.t. ☆ is an iso over $\left(\mathrm{Ran}^{untl}(X) \times \mathrm{Ran}^{untl}(X)\right)_{disj}$.

Stratawise description: $M \in D(\mathrm{Ran}^{\mathrm{untl}}(X))$ is $\in LFS(X)$

iff $\forall\, I = I_1 \sqcup I_2$, $f_i : I_i \longrightarrow I$

$$\Delta^!_{f_1}(M_{I_1}) \oplus \Delta^!_{f_2}(M_{I_2}) \xrightarrow{\nu(M)_{f_1} \oplus \nu(M)_{f_2}} M_I \quad \text{is } \simeq$$

when restricted to $(X^{I_1} \times X^{I_2})_{disj}$.

<u>Thm</u> $LFS(X)$ is the essential image of $i_!$.

<u>Pf</u> If $\mathcal{F} \in LFS(X)$ and $i^! \mathcal{F} = 0$, then $\mathcal{F}_{X^I}$ is zero on each strata of X^I, so $\mathcal{F}_{X^I} = 0$ $\forall I \Rightarrow \mathcal{F} = 0$.

For example,
$$\begin{tikzcd} \{*\} \arrow[r, shift left=2, "1"] \arrow[r, shift right=2, "2"'] & \{1,2\} \arrow[l, "g"'] \end{tikzcd}$$

$$\Delta^!_g(\mathcal{F}_{X^2}) \xrightarrow{\sim} \mathcal{F}_X = i^! \mathcal{F} = 0,$$

$$0 = \left(\Delta^!_1(\mathcal{F}_X) \oplus \Delta^!_2(\mathcal{F}_X)\right)\Big|_{X^2_{disj}} \xrightarrow{\sim} \mathcal{F}_{X^2}\Big|_{X^2_{disj}} .$$

Thus, since $i^!$ is conservative on $LFS(X)$,

it suffices to show that $\mathrm{im}(i_!) \subseteq LFS(X)$.

$$\begin{tikzcd}
 & X_{dR} \\
\left(\mathrm{Ran}_*^{\mathrm{untl}}(X) \times \mathrm{Ran}^{\mathrm{untl}}(X)\right)_{disj} \sqcup \left(\mathrm{Ran}^{\mathrm{untl}}(X) \times \mathrm{Ran}_*^{\mathrm{untl}}(X)\right)_{disj} \arrow[r, "U"] \arrow[d, "(q \times id) \sqcup (id \times q)"'] & \mathrm{Ran}_*^{\mathrm{untl}}(X) \arrow[u, "pr"'] \arrow[d, "q"] \\
\left(\mathrm{Ran}^{\mathrm{untl}}(X) \times \mathrm{Ran}^{\mathrm{untl}}(X)\right)_{disj} \arrow[r, "U"] & \mathrm{Ran}^{\mathrm{untl}}(X)
\end{tikzcd}$$

$$U^! \circ i_! = U^! \circ q_! \circ pr^! \simeq \left((q \times id)_! \boxplus (id \times q)_!\right) \circ U^! \circ pr^! \quad (\heartsuit)$$

But $pr \circ U = (pr \circ p_1) \sqcup (pr \circ p_2)$ so

$$(\heartsuit) = \left(p_1^! \circ q_! \circ pr^!\right) \oplus \left(p_2^! \circ q_! \circ pr^!\right) = (p_1^! \circ i_!) \oplus (p_2^! \circ i^!)$$

$\square$