Geometry of Bun_G

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1 What is Bun_G?

Let us briefly recall what a stack is on the fpqc site $\mathrm{Sch}_{\text{ét}}$.

Definition. A *prestack on* Sch is a pseudo-functor $\mathcal{F} : \mathrm{Sch}^{op} \to \mathrm{Gpd}$, i.e. it is the data of

- for every $X \in \mathrm{Sch}$ a groupoid $\mathcal{F}(X)$,
- for every morphism of schemes $f : X \to Y$ a functor $\mathcal{F}(f) : \mathcal{F}(Y) \to \mathcal{F}(X)$,
- for every $X \in \mathrm{Sch}_{\text{ét}}$ a natural isomorphism

  $$\alpha_X : \mathrm{id}_{\mathcal{F}(X)} \cong \mathcal{F}(\mathrm{id}_X),$$

  and
- for every pair of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ a natural isomorphism

  $$\beta_{f,g} : \mathcal{F}(g \circ f) \cong \mathcal{F}(f) \circ \mathcal{F}(g)$$

  of functors $\mathcal{F}(Z) \to \mathcal{F}(X)$,

satisfying obvious compatibility conditions (which are too tedious to write out).

Definition. A *stack* on $\mathrm{Sch}_{\text{ét}}$ is a prestack $\mathcal{F}$ on Sch satisfying the following glueing conditions for any étale cover $\{X_i \to X\}$:

- (Glueing Objects) Given an object $x_i \in \mathrm{ob}(\mathcal{F}(X_i))$ and morphisms $\phi_{i,j} : x_i|_{X_{i,j}} \to x_j|_{X_{i,j}}$ satisfying the cocycle condition

  $$\phi_{i,j}|_{X_{i,j,k}} \circ \phi_{j,k}|_{X_{i,j,k}} = \phi_{i,k}|_{X_{i,j,k}}$$

  (aka a descent datum for $\mathcal{F}$ with respect to the cover $\{X_i \to X\}$), there exists an object $x \in \mathrm{ob}\mathcal{F}(X)$ and isomorphisms $\phi_i : x|_{X_i} \cong x_i \in \mathcal{F}(X_i)$ such that $\phi_{j,i} \circ \phi_i|_{X_{i,j}} = \phi_j|_{X_{i,j}}$ (i.e. the descent datum is effective).
- (Glueing Morphisms) For any $x, y \in \mathcal{F}(X)$ and any morphisms $\phi_i : x|_{X_i} \to y|_{X_i}$ such that $\phi_i|_{X_{i,j}} = \phi_j|_{X_{i,j}}$, there exists a unique morphism $\phi : x \to y$ in $\mathcal{F}(X)$ such that $\phi|_{X_i} = \phi_i$ (i.e. the presheaf $(Y \to X) \mapsto \mathrm{Hom}_{\mathcal{F}(Y)}(x|_Y, y|_Y)$ on $\mathrm{Sch}_{X,\text{ét}}$ is a sheaf).

Here, we denote $X_{i,j} := X_i \times_X X_j$ and $X_{i,j,k} := X_i \times_X X_j \times_X X_k$ as usual.

Remark 1.1. Since we may view sets as groupoids in which the only morphisms are identity morphisms, we can view sheaves on Sch_{fpqc} as stacks as well. In particular, any scheme can be viewed as a stack via its functor of points.

Remark 1.2. We will also work with stacks on the étale site of Sch_S for some base scheme S; the above definitions obviously generalise.

The following examples are going to be very important for us.

Example 1.3. Let G be an algebraic group over a field k (i.e. a smooth affine group scheme of finite type over k). A *principal G-bundle on a k-scheme T* (aka a *G-torsor over T*) is a k-scheme P equipped with a faithfully flat map $\pi : P \to T$ (of k-schemes) and a G-action $\sigma : G \times_k P \to P$ such that $(\mathrm{proj}_2, \sigma) : G \times_k P \to P \times_T P$ is an isomorphism of schemes.

We now define the stack BG as the classifying stack of principal G-bundles, i.e. for any $T \in \mathrm{Sch}_k$ we set

$$\mathrm{ob}(BG(T)) = \{\text{principal } G\text{-bundles on } T\}$$
$$\mathrm{Hom}_{BG(T)}(P, P') = \{G\text{-equivariant maps } P \to P' \text{ over } T\}.$$

That this actually defines a stack is easy enough to check directly, using fpqc descent for morphisms. However, one can also check that BG is the sheafification of the naïve quotient prestack

$$(*/G)^{\text{naive}} : S \mapsto */G(S),$$

where $*/G(S)$ denotes the groupoid with a single object $*$ and whose automorphism group is $G(S)$. Thus $BG \simeq [*/G]$ is a quotient stack. This viewpoint will be useful later.

Example 1.4. Given a k-scheme X and a stack $\mathcal{Y}$ on $\mathrm{Sch}_{k,\text{ét}}$, we can define the stack $\underline{\mathrm{Hom}}(X, \mathcal{Y})$ whose value on a k-scheme T is the groupoid

$$\underline{\mathrm{Hom}}(X, \mathcal{Y})(T) := \mathcal{Y}(X \times_k T) = \mathrm{Hom}(X \times_k T, \mathcal{Y}).$$

It is straightforward to check directly that this satisfies the sheaf axioms since $\mathcal{Y}$ does.

Example 1.5. Given morphisms of prestacks $f : \mathcal{F} \to \mathcal{H}$ and $g : \mathcal{G} \to \mathcal{H}$, we can define their fibre product $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ as follows:

$$\mathrm{ob}(\mathcal{F} \times_{\mathcal{H}} \mathcal{G}(T)) := \{(x, y, \gamma) : x \in \mathrm{ob}(\mathcal{F}(T)), y \in \mathrm{ob}(\mathcal{G}(T)), \gamma : f(x) \cong g(y) \in \mathcal{H}(T).\}$$

$$\mathrm{Hom}_{\mathcal{F}\times_{\mathcal{H}}\mathcal{G}(T)}\big((x_1, y_1, \gamma_1), (x_2, y_2, \gamma_2)\big) := \left\{ (\phi, \psi) \;\middle|\; \begin{array}{l} \phi \in \mathrm{Hom}_{\mathcal{F}(T)}(x_1, x_2), \\ \psi \in \mathrm{Hom}_{\mathcal{G}(T)}(y_1, y_2), \end{array} \quad \begin{array}{ccc} f(x_1) & \xrightarrow[\cong]{f(\phi)} & f(x_2) \\ \cong\big\downarrow\gamma_1 & & \cong\big\downarrow\gamma_2 \\ g(y_1) & \xrightarrow[\cong]{g(\psi)} & g(y_2) \end{array} \right\}$$

If $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are stacks (resp. sheaves, resp. schemes), then so is $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$.

We now come to our main example.

Example 1.6. Given a smooth projective curve X over a field k, we define

$$\mathrm{Bun}_{G,X} := \underline{\mathrm{Hom}}(X, BG).$$

Given a test scheme T, by unwinding definitions we see that $\mathrm{Bun}_{G,X}(T) = BG(X \times_k T)$ is the groupoid of all principal G-bundles on $X \times_k T$ (i.e. families of G-bundles on X parametrized by T).

In order to define an Artin stack (aka an algebraic stack), we need the following definitions.

Definition. A morphism of prestacks $\mathcal{X} \to \mathcal{Y}$ is *representable by schemes* if for every morphism $T \to \mathcal{Y}$ from a scheme, the fibre product $\mathcal{X} \times_{\mathcal{Y}} T$ is a scheme.

An *algebraic space* is a sheaf X on $\mathrm{Sch}_{\text{ét}}$ such that there exists a scheme U and a surjective étale morphism $U \to X$ representable by schemes.

A morphism of prestacks $\mathcal{X} \to \mathcal{Y}$ is *representable by algebraic spaces* if for every morphism $T \to \mathcal{Y}$ from an algebraic space, the fibre product $\mathcal{X} \times_{\mathcal{Y}} T$ is an algebraic space.

An *Artin stack* is a stack $\mathcal{X}$ on $\mathrm{Sch}_{\text{ét}}$ such that there exists a scheme X and a surjective smooth morphism $f : X \to \mathcal{X}$ representable by algebraic spaces.

Here, it is worth noting what it means for a map of stacks $f : \mathcal{X} \to \mathcal{Y}$ to have certain properties. Given a property $\mathscr{P}$ enjoyed by morphisms of schemes that is stable under base change and étale local on the source (e.g. surjectivity, smoothness, étaleness, locally of finite presentation, etc), we say that a map $\mathcal{X} \to \mathcal{Y}$ representable by algebraic spaces has property $\mathscr{P}$ if for all schemes $T \to \mathcal{Y}$ and any étale presentation $U \to \mathcal{X} \times_{\mathcal{Y}} T$ by a scheme U, the composition $U \to \mathcal{X} \times_{\mathcal{Y}} T \to T$ (now a morphism of schemes) has property $\mathscr{P}$.

Schemes are examples of algebraic spaces, and algebraic spaces are examples of Artin stacks.

2 $\mathrm{Bun}_{\mathrm{GL}_n,X}$ is an Artin stack

Let us now show that $\mathrm{Bun}_{\mathrm{GL}_n,X}$ is an Artin stack whenever X is a smooth projective curve over k. First, note that GL_n bundles on S are equivalent to rank n vector bundles on S (locally free $\mathcal{O}_S$-modules), since they are both determined by the same descent data (a trivialising cover $\{S_i \to S\}$ and for each i, j an element of $GL_n(S_i \times_S S_j)$). We now sketch a proof of the following result.

Proposition 2.1. $\mathrm{Bun}_{\mathrm{GL}_n,X}$ *is an Artin stack.*

We now recall some algebraic geometry. Suppose S is a projective k-scheme and $\mathcal{O}_S(1)$ is some fixed ample line bundle. Then, Serre's vanishing theorem says that for any coherent $\mathcal{O}_S$-module $\mathcal{V}$, there exists r large enough such that $\mathcal{V}(r) := \mathcal{V} \otimes_{\mathcal{O}_S} \mathcal{O}_S(1)^{\otimes r}$ is generated by global sections, i.e.

$$H^0(S, \mathcal{V}(r)) \otimes \mathcal{O}_S \twoheadrightarrow \mathcal{E}(r),$$

and $H^i(S, \mathcal{V}(r)) = 0$ for $i \geq 1$. Notice that $H^0(S, \mathcal{V}(r)) \otimes \mathcal{O}_S = \pi^*\pi_*\mathcal{V}(r)$ for $\pi : S \to \operatorname{Spec} k$ the structure map. Next, notice that for any vector bundle $\mathcal{V} \in \mathcal{U}_r(T)$ by the Riemann-Roch theorem

$$\dim H^0(X_t, \mathcal{V}_t(r)) = \deg \mathcal{V}_t(r) + \operatorname{rank}(\mathcal{V}_t(r))(g-1) = n \deg \mathcal{O}_X(r) + \deg \mathcal{V}_s + n(g-1)$$

where g is the genus of X.

Let $\mathcal{U}_r$ be the moduli stack of such vector bundles, i.e.

$$\mathcal{U}_r(T) := \{\mathcal{V} \in \operatorname{Bun}_{\mathrm{GL}_n,X}(T) : R^i p_{T*}\mathcal{V}(r) = 0 \text{ for all } i \geq 1 \text{ and } p_T^* p_{T*}\mathcal{V}(r) \to \mathcal{V}(r) \text{ is surjective}\}.$$

Serre's theorem then tells us that $\operatorname{Bun}_{\mathrm{GL}_n,X} = \bigcup_{r \geq 0} \mathcal{U}_r$.

Lemma 2.2. *$\mathcal{U}_r$ is an open subfunctor of* $\operatorname{Bun}_{\mathrm{GL}_n,X}$.

Proof Sketch. We need to check that for every scheme T, the fibre product $\mathcal{U}_r \times_{\operatorname{Bun}_{\mathrm{GL}_n,X}} T$ is an open subscheme of T. This is the same as asking that, given any vector bundle $\mathcal{V}$ on $X \times_k T$, the set of points $t \in T$ such that $\mathcal{V}_t \in \mathrm{U_r}(X \times_k t)$ is open in T. That $H^i(X_s, \mathcal{V}_s(r)) = 0$ is an open condition is clear, since it is the complement of the support of $R^i p_{T,*}\mathcal{V}(r)$. Asking that $\mathcal{V}_t(r)$ is globally generated is also an open condition, since the natural map $p_T^* p_{T*}\mathcal{V} \to \mathcal{V}$ will be surjective on fibres over an open subset of T. □

Now, given a coherent $\mathcal{O}_X$-module $\mathcal{E}$ and a polynomial $P \in \mathbb{Q}[x]$, Grothendieck showed that the functor

$$\operatorname{Quot}^P_{X/k}(\mathcal{E}) : \mathrm{Sch}_k^{op} \to \mathrm{Set}, T \mapsto \left\{ (\mathcal{F}, q) \,\middle|\, \begin{array}{c} \mathcal{F} \text{ a finitely presented quasi-coherent } \mathcal{O}_{X \times_k T}\text{-module} \\ \mathcal{F} \text{ flat over } T \text{ and Hilbert polynomial of } \mathcal{F}_t \text{ is } P \text{ for all } t \in T \\ q : \mathcal{E} \twoheadrightarrow \mathcal{F} \text{ a surjection} \end{array} \right\}$$

is a projective k-scheme, so that the functor

$$\operatorname{Quot}_{X/k}(\mathcal{E}) : \mathrm{Sch}_k^{op} \to \mathrm{Set}, T \mapsto \left\{ (\mathcal{F}, q) \,\middle|\, \begin{array}{c} \mathcal{F} \text{ a finitely presented quasi-coherent } \mathcal{O}_{X \times_k T}\text{-module flat over } T \\ q : \mathcal{E} \twoheadrightarrow \mathcal{F} \text{ a surjection} \end{array} \right\}$$

is a disjoint union of projective k-schemes. In particular, it is a k-scheme. Thus, the open subfunctor

$$\operatorname{Quot}^{lf}_{X/k}(\mathcal{E}) : \mathrm{Sch}_k^{op} \to \mathrm{Set}, T \mapsto \left\{ (\mathcal{F}, q) \,\middle|\, \begin{array}{c} \mathcal{F} \text{ a vector bundle on } X \times_k T \text{ (so flat over } T) \\ q : \mathcal{E} \twoheadrightarrow \mathcal{F} \text{ a surjection} \end{array} \right\}$$

of $\operatorname{Quot}_{X/k}(\mathcal{E})$ is a k-scheme. It is then clear that

$$Y_{d,r}(T) := \left\{ (\mathcal{F}, q, \alpha) \,\middle|\, \begin{array}{c} (\mathcal{F}, q) \in \operatorname{Quot}^{lf}_{X/k}\left(\mathcal{O}_X(-r)^{\oplus(n \deg \mathcal{O}_X(r) + d + n(g-1))}\right)(T) \\ \alpha : \mathcal{O}_T(-r)^{\oplus(n \deg \mathcal{O}_X(r) + d + n(g-1))} \cong p_{T*}\mathcal{F} \end{array} \right\}$$

is again an open subscheme of the Quot scheme, and thus in particular is a scheme itself.

By the discussion preceding the lemma, we see that we have a surjection

$$Y_r := \bigcup_{d \in \mathbb{Z}} Y_{d,r} \twoheadrightarrow \mathcal{U}_r$$

from a scheme Y_r onto $\mathcal{U}_r$. Thus we have represented $\operatorname{Bun}_{\mathrm{GL}_n}$ as a quotient of a scheme. Moreover this quotient is quite well-structured.

Lemma 2.3. *$Y_r \to \mathcal{U}_r$ is a $\mathrm{GL}_{\Phi(r)}$ torsor, where $\Phi(r) := n \deg \mathcal{O}_X(r) + d + n(g-1)$*

Proof Sketch. Consider any map $S \to \mathcal{U}_r$. This is an object of $\mathcal{U}_r(S)$, so is actually a vector bundle $\mathcal{V}$ on $X \times_k S$ satisfying various conditions. Lifting this to Y_r is simply picking an isomorphism $\mathcal{O}_S^{\Phi(r)} \cong p_{S*}\mathcal{V}(r)$. The set of such identifications is quite obviously a $\mathrm{GL}_{\Phi(r)}(S)$-torsor. □

This quotient being a principal $\mathrm{GL}_{\Phi(r)}$ bundle in particular implies that the map $Y_r \to \mathcal{U}_r$ is smooth (since $\mathrm{GL}_{\Phi(r)}$ is itself smooth). This completes the proof that $\operatorname{Bun}_{\mathrm{GL}_n,X}$ is an Artin stack.

3 $\mathrm{Bun}_{G,X}$ is an Artin stack

We now can prove that $\mathrm{Bun}_{G,X}$ is an Artin stack, by reducing to the GL_n case. To simplify notation, we suppress the X. Pick any injection $G \hookrightarrow \mathrm{GL}_n$ (faithful representations always exist).

Proposition 3.1. *For any injection of algebraic groups $H \hookrightarrow G$, the induced map $\mathrm{Bun}_H \to \mathrm{Bun}_G$ is representable by schemes.*

The reason this lemma is useful is because of the following easy result.

Lemma 3.2. *If $\mathcal{X} \to \mathcal{Y}$ is representable by schemes and $\mathcal{Y}$ is an algebraic space (resp. Artin stack), then $\mathcal{X}$ is also an algebraic space (resp. Artin stack).*

We thus deduce immediately the following.

Corollary 3.2.1. Bun_G *is an Artin stack.*

Let us now try to prove the proposition. First, we describe the induced map $\mathrm{Bun}_H \to \mathrm{Bun}_G$. Recall that $\mathrm{Bun}_G = \underline{\mathrm{Hom}}(X, BG)$ and $\mathrm{Bun}_H = \underline{\mathrm{Hom}}(X, BH)$. The map $BH \to BG$ sends a T-point $\pi : P \to T$ of BH to the quotient sheaf $(G \times_k P)/H$; that this is a scheme follows by looking at a trivialization - when P is trivial, so $P \simeq H$, then $(G \times_k P)/H \simeq G$.

Lemma 3.3. *$BH \to BG$ is quasi-projective and representable by schemes.*

Proof Sketch. Suppose we have a morphism $T \to BG$; we want to show that $BH \times_{BG} T$ is a scheme. The morphism $T \to BG$ gives a principal G-bundle $\pi : P \to T$. We claim that $BH \times_{BG} T$ is the scheme $(H\backslash G \times_k P)/G$. Here, $H\backslash G$ exists as a quotient in schemes since we are working over a field[1]. Indeed, suppose given a morphism $S \to BH$ (i.e. a principal H-bundle $\rho : Q \to S$) and $f : S \to T$ such that we have a commuting square[2].

$$\begin{array}{ccc} S & \xrightarrow{\ f\ } & T \\ \downarrow{\scriptstyle Q} & & \downarrow{\scriptstyle P} \\ BH & \longrightarrow & BG \end{array}$$

That this commutes is simply asking that $P_S := S \times_T P = (Q \times_k G)/H$ as principal G-bundles over S. Thus we have G-equivariant morphisms $P_S \to H\backslash G$ and $P_S \to P$ over k, so that we have a G-equivariant morphism $Q' \to (H\backslash G) \times_k P$. However, the data of a morphism $S \to (H\backslash G \times_k P)/G$ is exactly the data of a principal G-bundle $Q' \to S$ and a G-equivariant morphism $Q' \to H\backslash G \times_k P$.

That $BH \to BG$ is quasi-projective follows from the fact that $H\backslash G$ is quasi-projective. □

Proposition 3.1 then follows from the following result.

Proposition 3.4. *Suppose $\mathcal{Y} \to \mathcal{Z}$ is a quasi-projective map of prestacks that is representable by schemes. Let X be projective over k. Then, the induced map $\underline{\mathrm{Hom}}(X, \mathcal{Y}) \to \underline{\mathrm{Hom}}(X, \mathcal{Z})$ is representable by schemes.*

Proof Sketch. Fix an S-point of $\underline{\mathrm{Hom}}(X, \mathcal{Z})$, i.e a map $S \times_k X \to \mathcal{Z}$. Unravelling definitions, the fibre product $\underline{\mathrm{Hom}}(X, \mathcal{Y}) \times_{\underline{\mathrm{Hom}}(X,\mathcal{Z})} S$ is the prestack over S that sends an S-scheme S' to

$$\mathrm{Hom}_{S\times_k X}\big(S' \times_k X, \mathcal{Y} \times_{\mathcal{Z}} (S \times_k X)\big)$$

where note that $Y_S := \mathcal{Y} \times_{\mathcal{Z}} (S \times_k X)$ is a quasi-projective scheme over $S \times_k X$ by assumption. Set $Y := \mathcal{Y} \times_{\mathcal{Z}} X$. Then, the above fibre product coincides with the so-called sections functor

$$\mathrm{Sect}_S(X, Y) : \mathrm{Sch}^{op}_{S,\text{ét}} \to \mathrm{Set}, \quad S' \mapsto \mathrm{Hom}_{X\times_k S'}(X \times_k S', Y \times_k S') = \mathrm{Hom}_{X\times_k S}(X \times_k S', Y \times_k S).$$

It is a fact that the sections functor $\mathrm{Sect}_S(X, Y)$ is a scheme if X is projective and $Y \to X$ is quasi-projective. As this is true in our case, the proposition follows. □

[1] such a quotient is not necessarily a scheme over an arbitrary base!

[2] This is a 2-commuting square, which means we need to remember the isomorphism between the two maps; let's just sweep that under the rug.

4 $\mathrm{Bun}_{G,X}$ is a smooth Artin stack

We now want to check smoothness. For this, we need to use the infinitesimal criterion for smoothness.

Theorem 4.1. *Suppose $\mathcal{X}$ is an Artin stack locally of finite type over k. Then $\mathcal{X}$ is smooth over k if and only if for all surjections of k-algebras $A \to A_0$ with square-zero kernel (i.e. $A_0 = A/I$ and $I^2 = 0$) fitting into the following diagram of solid arrows,*

$$\begin{array}{ccc} \operatorname{Spec} A_0 & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow & \\ \operatorname{Spec} A & & \end{array}$$

there exists a lift $\operatorname{Spec} A \to \mathcal{X}$ making the entire diagram commutative.

Thus, we have reduced to a deformation-theoretic argument.

Proposition 4.2. Bun_G *is smooth over k.*

Proof Sketch. Fix a faithful representation $G \hookrightarrow \mathrm{GL}_n$. We defined open substacks $\mathcal{U}_{\mathrm{GL}_n,r}$ of $\mathrm{Bun}_{\mathrm{GL}_n}$ when proving algebraicity; let

$$\mathcal{U}_r := \mathcal{U}_{\mathrm{GL}_n,r} \times_{\mathrm{Bun}_{\mathrm{GL}_n}} \mathrm{Bun}_G.$$

We use the infinitesimal criterion of smoothness for the stack $\mathcal{U}_r$. Our description of $\mathcal{U}_{\mathrm{GL}_n,r}$ as the quotient of an open subscheme of a Quot scheme by GL_* tells us that $\mathcal{U}_{\mathrm{GL}_n,r}$, and thus $\mathcal{U}_r$, is locally of finite presentation.

Pick any surjection of k-algebras $A \to A_0$ with square-zero kernel I, and let $\mathcal{V}$ be a A_0-point of Bun_G (equivalently, a morphism $\operatorname{Spec} A_0 \to \mathrm{Bun}_G$). Thus $\mathcal{V}_0$ is a G-bundle on $X_{A_0} := X \times_k \operatorname{Spec} A_0$, and we want to lift it to a G-bundle on $X_A := X \times_k \operatorname{Spec} A$. Now, $\mathcal{V}_0$ corresponds to a map $X_{A_0} \to BG$, so we can pull-back the quasi-coherent sheaf on BG corresponding to the adjoint representation on $\mathfrak{g} = \mathrm{Lie}(G)$ to get a sheaf $\mathfrak{g}(\mathcal{V}_0)$ on X (for example, for $G = \mathrm{GL}_n$, we have $\mathfrak{g}(\mathcal{V}_0) = \underline{\mathrm{End}}_X(\mathcal{V}_0)$). Deformation theory then tells us there exists an element $\mathrm{ob} \in H^2(X, \mathfrak{g}(\mathcal{V}_0))$ such that $\mathrm{ob} = 0$ if and only if there exists an extension $\mathcal{V}$ of $\mathcal{V}_0$ to X_A. However, X is a (smooth projective) curve over k, so that H^2 vanishes. Hence we can always lift. □