Overview Talk

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Abstract

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1 Overview

Let X be a connected smooth proper curve over a field of characteristic zero and let G be an affine algebraic group. This data specifies a moduli space $\mathrm{Bun}_G(X)$ which parameterizes principal G-bundles on X. Our goal this semester is to study the category of $\mathcal{D}$-modules on this moduli space.

Motivation What do we hope to gain from this? If X were instead defined over $\mathbf{F}_p$, then principal G-bundles on X have the following description: given such a bundle $\mathcal{E}$, we can trivialize $\mathcal{E}$ locally on an affine open U[1]. Since X is a curve, the complement of U is a finite set of points $x_1, \ldots, x_n$. We can also trivialize $\mathcal{E}$ on the formal disks D_i around each x_i. The trivialization on U and the trivialization on D_i are related by a transition map $g_i \in G(\mathcal{K}_i)$, where $\mathcal{K}_i$ is the ring of functions of the punctured formal disk $\mathring{D}_i$. These transition maps assemble to give an element $g \in G(\mathbf{A}_X)$ (here $\mathbf{A}_X$ is the ring of adeles associated to X):

$$g_x = \begin{cases} g_i & x = x_i \\ 1 & \text{otherwise} \end{cases}$$

Of course, the element g that we produced depended on our choice of U, our choice of trivialization of $\mathcal{E}$ on U, and our choices of trivializations of $\mathcal{E}$ on each formal disk D_i. However, the class of g in $G(K_X)\backslash G(\mathbf{A}_X)/G(\mathcal{O}_X)$ is independent of these choices (here K_X is the field of fractions of X and $\mathcal{O}_X$ is the ring of integral adeles). In fact, something stronger is true:

Theorem 1.1. *The points of* $\mathrm{Bun}_G(X)$ *are in bijection with* $G(K_X)\backslash G(\mathbf{A}_X)/G(\mathcal{O}_x)$[2].

We can intepret this theorem (called "Weil Uniformization") as saying that $\mathrm{Bun}_G(X)$ gives an algebro-geometric description of the double coset space $G(K_X)\backslash G(\mathbf{A}_X)/G(\mathcal{O}_x)$. When G is reductive, the space $G(K_X)\backslash G(\mathbf{A}_X)/G(\mathcal{O}_x)$ is the domain of definition for unramified automorphic functions for the field K_X, so this result opens up the possibility of understanding the theory of automorphic functions (at least for function fields) via algebraic geometry.

More concretely, given an ℓ-adic sheaf $\mathcal{F}$ on $\mathrm{Bun}_G(X)$, the fiber of $\mathcal{F}$ at any point $x \in \mathrm{Bun}_G(X)$ carries an action of the Galois group $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$, with associated character χ_x. We can associate an unramified automorphic function

$$f_{\mathcal{F}} : G(K_X)\backslash G(\mathbf{A}_X)/G(\mathcal{O}_x) \to \mathbf{Q}_\ell$$

to $\mathcal{F}$ by setting

$$f_{\mathcal{F}}(x) = \chi_x(\mathrm{Frob}).$$

In this way, the theory of ℓ-adic sheaves on $\mathrm{Bun}_G(X)$ "geometrizes" the theory of unramified automorphic functions.

[1] If G isn't simply connected, this may only be possible after extending coefficients to $\mathbf{F}_q$.

[2] Because of the issue mentioned in the first footnote, when G isn't simply connected $G(K_X)\backslash G(\mathbf{A}_X)/G(\mathcal{O}_x)$ has to be interpreted as a the points of a quotient of etale sheaves.

Although we've shown that the study of ℓ-adic sheaves on $\mathrm{Bun}_G(X)$ for X a curve in positive characteristic is well-motivated by relations to the classical theory of automorphic forms, the reader is justified in wondering what relationship this has to the study of $\mathcal{D}$-modules on $\mathrm{Bun}_G(X)$ for X a curve in characteristic zero. At the moment, the relationship is still conjectural, but the idea is that every smooth curve over $\mathbf{F}_p$ can be extended to a smooth curve over $\mathbf{Z}_p$, and this allows us to interpolate between characteristic zero and characteristic p. In characteristic zero, there are standard techniques that allow for comparisons between constructible ℓ-adic sheaves and regular holonomic $\mathcal{D}$-modules.

The benefit we gain by studying $\mathcal{D}$-modules instead of ℓ-adic sheaves is that $\mathcal{D}$-modules are of a more algebraic nature: $D(\mathrm{Bun}_G(X))$ is monadic over $\mathrm{QCoh}(\mathrm{Bun}_G)$, which intuitively means that $\mathcal{D}$-modules on $\mathrm{Bun}_G(X)$ are "defined by generators and relations". This fact will allow us to produce $\mathcal{D}$-modules on Bun_G from representations of affine Kac-Moody algebras through a procedure called **localization**.

Hecke Eigensheaves So far we've been intentionally vague about what we mean by "studying $\mathcal{D}$-modules on $\mathrm{Bun}_G(X)$". In the classical theory of modular forms, there is a distinguished class of modular forms which are especially well-behaved: the Hecke-eigenforms. These modular forms have the nice property that their associated L-functions admit an Euler product expansion.

Definition 1.2. The **Hecke operator** T_p is an endomorphism of the space of modular forms of a fixed weight k. Given such a modular form f, thought of as a function on the moduli space of lattices,

$$T_p f(L) = \sum_{\substack{L' \subseteq L \\ [L:L']=p}} f(L')$$

Definition 1.3. A **Hecke eigenform** is an eigenvector for all of the Hecke operators.

A standard result is that the space of unramified modular forms is spanned by the Hecke eigenforms. Our goal for this semester is essentially to geometrize this result, by showing that there is a collection of $\mathcal{D}$-modules $\{\mathcal{F}_i\}$, with each $\mathcal{F}_i$ satisfying an analogue of the Hecke eigenform condition, such that the set $\{\mathcal{F}_i\}$ generates the category $D(\mathrm{Bun}_G(X))$ [3].

Let's now try to indicate what the analogue of the Hecke eigenform condition is in the geometric setting. Note that if we have a lattice L and a sublattice L' with the property that $[L : L'] = p$, then the lattices become equal (as subsets of $\mathbf{C}$) after inverting p. Set

$$\mathcal{H}ecke_{(p)} = \left\{(L, L') \mid L[p^{-1}] = L'[p^{-1}]\right\}.$$

There is a special subset Z_p of $\mathcal{H}ecke_{(p)}$ consisting of pairs $z = (L, L')$ with $[L : L'] = p$. Z_p has two maps π_1, π_2 to the set of sublattices of $\mathbf{C}$ (projection onto L and L').

$$T_p f(L) = \sum_{\substack{z \in Z_p \\ \pi_1(z)=L}} f(\pi_2(z)) = \pi_{1,*}\pi_2^* f(L).$$

Roughly speaking, by analogy, in the geometric setting, we will produce some subspace Z of the space

$$\mathcal{H}ecke_x = \left\{(\mathcal{E}, \mathcal{E}') \mid \mathcal{E}|_{X\setminus x} \simeq \mathcal{E}'|_{X\setminus x}\right\}$$

so that the **geometric Hecke operators** are given by

$$H_x \mathcal{F} = \pi_{1,*}\pi_2^! \mathcal{F}.$$

We will try to produce **Hecke eigensheaves**, which are sheaves $\mathcal{F}$ so that

$$H_x \mathcal{F} = E_x \otimes \mathcal{F}$$

for some fixed local system E on X.

[3] We will fall short of this goal.

Localization We previously noted that an important benefit of $D(\mathrm{Bun}_G)$ is that it is monadic over $\mathrm{QCoh}(\mathrm{Bun}_G)$, which means that a $\mathcal{D}$-module can be thought of as "a quasi-coherent sheaf with extra structure". As a first step towards constructing $\mathcal{D}$-modules on Bun_G, let's outline a way to construct quasi-coherent sheaves on Bun_G.

For a space Y, the arc space $\mathcal{L}^+Y$ is the space of maps from the formal disk into Y. When Y is a group G the arc space is also a group (where multiplication is defined "pointwise" on D). Now suppose we choose a point x on our given curve X, as well as a local coordinate t at x. This data gives an identification of the formal neighborhood of x with the formal disk. Given a principal G-bundle $\mathcal{E}$ on X, restricting $\mathcal{E}$ to the formal neighborhood of x gives a principal $\mathcal{L}^+G$-bundle[4], and this defines a map $\varphi_x : \mathrm{Bun}_G(X) \to B\mathcal{L}^+G$.

Pulling back along this map gives a functor $\varphi_x^* : \mathrm{QCoh}(B\mathcal{L}^+G) \to \mathrm{QCoh}(\mathrm{Bun}_G)$, which gives us a way to produce quasi-coherent sheaves on Bun_G from representations of $\mathcal{L}^+G$.

[4]I actually don't know a reference for this! The claim is that for any principal G-bundle on $\mathrm{Spec}\, A[\![t]\!]$ there is an etale cover of U of $\mathrm{Spec}\, A$ such that G is trivial on $U[\![t]\!]$. I think this should follow from something like the infinitesimal lifting criterion and Artin algebraization.